seit 1558

Expected Discrepancies

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der
Fakultät für Mathematik und Informatik
der Friedrich-Schiller-Universität Jena

von Dipl.-Math. Heidi Weyhausen,
geboren am 23.09.1986 in Saalfeld

Gutachter:

1. Univ.Prof. Dr. Aicke Hinrichs, Johannes Kepler Universität Linz
2. Prof. Dr. Erich Novak, Friedrich-Schiller-Universität Jena
3. A.Univ.Prof. Dr. Friedrich Pillichshammer, Johannes Kepler Universität Linz

Tag der öffentlichen Verteidigung: 02. Juli 2015

Acknowledgments

I owe my deepest gratitude to Prof. Dr. Aicke Hinrichs not only for supporting me with his expertise but also for being patient and sympathetic with me for the last five years. I am also very grateful to Prof. Dr. Erich Novak for helpful discussions and suggestions. I would like to thank Andreas Weyhausen for supporting me in doing the numerical simulations. I am thankful for having met many nice people while working in the Mathematical Institute in Jena. Thank you DI Dr. Christoph Aistleitner, Prof. Dr. Dmitriy Bilyk, Dr. Toralf Burghoff, JProf. Dr. Henning Kempka, Isabelle Kuhwald, Robert Kunsch, Dr. Lev Markhasin, Dr. Christian Richter, Dr. Daniel Rudolf, Dr. Philipp Rudolph, Prof. Dr. Winfried Sickel, Dr. Mario Ullrich, Dr. Markus Weimar and Prof. Dr. Henryk Woźniakowski for many interesting discussions and insights.
Finally, i would like to thank my family for keeping me grounded. Thank you Mama, Papa, Andreas, Annett and Jona.

Contents

Zusammenfassung

Während die Forschung in der Diskrepanztheorie heute verschiedenste Zweige der Mathematik betrifft, hatte sie ihre Anfänge in der Theorie der Gleichverteilung mit einer grundlegenden Arbeit von Weyl von 1916 ([53]) und einer Vermutung von van der Corput von 1935 ([11, 12]):

Vermutung. *Sei $s_1, s_2, s_3, \ldots$ eine unendliche Folge von reellen Zahlen zwischen 0 und 1. Dann existiert für jedes beliebig große κ eine natürliche Zahl n und zwei Teilintervalle von $(0,1)$ gleicher Länge, sodass die Anzahl der s_ν $(\nu = 1, \ldots, n)$, die in einem der Teilintervalle liegen von der Anzahl derjenigen s_ν, die im zweiten Teilintervall liegen, um mehr als κ abweicht.*

Diese Vermutung, laut der eine Folge niemals perfekt gleichverteilt sein kann, wurde 1945 von van Aardenne-Ehrenfest bewiesen ([3]).

Im Jahr 1954 veröffentlichte Roth einen Meilenstein der Diskrepanztheorie ([43]). Er formulierte das obige Problem um und verwendete dafür die Diskrepanzfunktion. In dieser Arbeit definieren wir die Diskrepanzfunktion als

$$D(B, \mathcal{P}) = \frac{1}{N} \sum_{j=1}^{N} \chi_B(t_j) - \mathrm{vol}(B)$$

für ein $B = [0, x] \subset [0,1]^d$ und eine Punktmenge $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$. Roth zeigte, dass

$$\int_{[0,1]^2} (N D(B, \mathcal{P}))^2 \, \mathrm{d}x > c \log N$$

für jede Punktmenge $\mathcal{P} \subset [0,1]^2$ gilt. Dieses Resultat war der Startpunkt intensiver Forschung im Bereich der Diskrepanztheorie. Einen guten historischen Überblick, vom Beginn bis in die 1980er, erhält man zum Beispiel in [4].

Üblicherweise interessiert man sich für die Norm der Diskrepanzfunktion in gewissen Funktionenräumen. In dieser Arbeit untersuchen wir den klassischen Fall der L_p-Normen für $0 < p \leq \infty$. Für $0 < p < \infty$ definieren wir die L_p-Stern-Diskrepanz

$$D_p^*(\mathcal{P}) = \left(\int_{[0,1]^d} |D(B, \mathcal{P})|^p \, \mathrm{d}x \right)^{1/p}$$

und für $p = \infty$ die Stern-Diskrepanz

$$D^*_\infty(\mathcal{P}) = \sup_{x\in[0,1]^d} |D(B,\mathcal{P})| .$$

Das Ziel dieser Arbeit ist die Erforschung erwarteter Diskrepanzen: Wir wählen eine zufällige Punktmenge $\mathcal{P}$ mit unabhängigen, gleichverteilten Punkten $t_1,\dots,t_N$ in $[0,1]^d$ und berechnen den Erwartungswert von L_p-Diskrepanzen.

In Kapitel 1 führen wir relevante Bezeichnungen ein, präsentieren Definitionen und Resultate aus der Wahrscheinlichkeitstheorie und geben eine kurze Einführung in die Diskrepanztheorie. Anschließend ist diese Arbeit dreigeteilt, mit - zum Großteil - unabhängigen Kapiteln 2, 3 und 4.

In Kapitel 2 analysieren wir erwartete L_p-Diskrepanzen für $0 < p < \infty$. Wir erweitern dabei den Diskrepanzbegriff zur L_p-B-Diskrepanz. Dies bedeutet, dass wir allgemeine Mengen B anstatt verankerter Boxen $[0,x]$ erlauben (siehe Definition 1.18). Wir definieren die durchschnittliche L_p-Stern-Diskrepanz als

$$\mathrm{av}^*_p(N,d) = \left(\mathbb{E}\left(D^*_p(\mathcal{P})^p\right)\right)^{1/p}.$$

Der Erwartungswert wird über Mengen $\mathcal{P}$ von N unabhängigen und gleichverteilten Punkten $t_1,\dots,t_N \in [0,1]^d$ berechnet. Außerdem definieren wir die durchschnittliche L_p-B-Diskrepanz als

$$\mathrm{av}^B_p(N,d) = \left(\mathbb{E}\left(D^B_p(\mathcal{P})^p\right)\right)^{1/p}.$$

Die erste Definition wurde von Heinrich, Novak, Wasilkowski und Woźniakowski in [30] eingeführt. Diese bewiesen für geradzahlige p die Gleichheit

$$\mathrm{av}^*_p(N,d)^p = \sum_{r=p/2}^{p-1} C(r,p,d)N^{-r},$$

mit bekannten Konstanten $C(r,p,d)$. Für beliebige p veröffentlichte Steinerberger in [50] ein Ergebnis für den Grenzwert in N, allerdings mit einer Lücke im Beweis. Das Hauptresultat von Kapitel 2 ist eine Verallgemeinerung des Resultates von Steinerberger für die durchschnittliche L_p-B-Diskrepanz, formuliert in Theorem 2.2:

Theorem. *Sei $p > 0$ und $d \in \mathbb{N}$, sei außerdem (Ω_d, μ_d) ein Wahrscheinlichkeitsraum und $\{B(x) : x \in \Omega_d\} \subset 2^{[0,1]^d}$ die erlaubten Mengen. Dann gilt*

$$\lim_{N\to\infty} N^{p/2}\mathrm{av}^B_p(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int\limits_{\Omega_d}\left[\lambda^d(B(x))\Big(1-\lambda^d(B(x))\Big)\right]^{p/2}\mathrm{d}\mu_d(x).$$

Dieses Resultat wurde in [35] veröffentlicht.

In Kapitel 3 untersuchen wir erwartete Stern-Diskrepanzen, das heißt $p = \infty$. Die beste bekannte obere Schranke für die minimale Stern-Diskrepanz wurde von Heinrich, Novak, Wasilkowski und Woźniakowski in [30] bewiesen. Sie zeigten

$$D^*(N,d) = \inf_{\substack{\mathcal{P}\subset[0,1]^d \\ \#\mathcal{P}=N}} D^*_\infty(\mathcal{P}) \leq c\sqrt{\frac{d}{N}}.$$

Es sind keine passenden unteren Schranken für $D^*(N,d)$ bekannt. Die beste bekannte untere Schranke wurde von Hinrichs in ([32]) gezeigt:

$$D^*(N,d) \geq \min\left\{\varepsilon_0, c\,\frac{d}{N}\right\}$$

mit Konstanten $c, \varepsilon_0 > 0$. Doerr fragte in [18] nach zufälligen statt den optimalen Punkten und bewies die Existenz einer Konstanten $K > 0$, sodass

$$\mathbb{E}D^*_\infty(\mathcal{P}) \geq K\sqrt{\frac{d}{N}}$$

erfüllt ist. Interessanterweise gilt für eine mit a gewichtete Diskrepanz für $d = N = 1$ die Ungleichung

$$\inf_a \mathbb{E}D^*_\infty(a,\mathcal{P}) < \mathbb{E}D^*_\infty(\mathcal{P}),$$

was uns zur Berechnung einer unteren Schranke für den gewichteten Fall motiviert. Dies liefert Theorem 3.3, das Hauptresultat aus Kapitel 3:

Theorem. *Es gibt eine Konstante $K > 0$, sodass folgende Aussage wahr ist: Seien $N, d \in \mathbb{N}$ mit $d \leq N$. Sei $\mathcal{P}$ eine Menge von N unabhängigen und gleichverteilten Punkten aus $[0,1]^d$. Sei außerdem $a = (a_j)_{j=1}^N$ eine Folge von Gewichten. Dann erfüllt der Erwartungswert der gewichteten Stern-Diskrepanz*

$$\mathbb{E}D^*_\infty(a,\mathcal{P}) \geq K\cdot\sqrt{\frac{d}{N}}.$$

Schließlich fragen wir in Abschnitt 3.3 nach dem Erwartungswert der gewichteten Stern-Diskrepanz mit optimalen statt beliebigen, festen Gewichten. Das heißt, wir wählen die Gewichte für eine feste Punktmenge so, dass die Diskrepanz minimal wird. Wir zeigen für Dimension $d = 1$ Theorem 3.6:

Theorem. *Der Erwartungswert der optimal gewichteten Stern-Diskrepanz in Di-*

mension $d = 1$ hat in N das asymptotische Verhalten

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \asymp \frac{\log N}{N}.$$

In Kapitel 4 kehren wir wieder zum Fall der L_2-Diskrepanz zurück. Aufgrund des berühmten Resultates von Roth ([43]) wissen wir, dass

$$D_2^*(\mathcal{P}) \geq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

für jede Punktmenge $\mathcal{P} \subset [0,1]^d$ mit N Punkten gilt. Im Jahr 2002 konnten Chen und Skriganov ([10]) als Erste eine explizite Punktmenge konstruieren, die

$$D_2^*(\mathcal{P}) \leq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

erfüllt. Bis zu diesem Zeitpunkt waren lediglich probabilistische Konstruktionen bekannt ([22, 46]). Aus der Arbeit von Heinrich, Novak, Wasilkowski und Woźniakowski ([30]) wissen wir, dass

$$\mathbb{E} D_2^*(\mathcal{P})^2 = C_d \frac{1}{N}$$

gilt. Dies bedeutet, dass zufällige Punkte eine schlechtere Ordnung in N für die L_2-Diskrepanz liefern als optimale Punkte. Inspiriert vom Resultat für die optimal gewichtete Stern-Diskrepanz aus Kapitel 3 berechnen wir den Erwartungswert der optimal gewichteten L_2-Diskrepanz und hoffen auf ein besseres asymptotisches Verhalten in N. Wir zeigen Theorem 4.4:

Theorem. *Für den Erwartungswert der optimal gewichteten L_2-Diskrepanz in Dimension $d = 1$ gilt*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \inf_{\substack{\mathcal{P} \subset [0,1] \\ \#\mathcal{P} = N}} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{1}{N^2}.$$

Dieses Theorem lässt uns hoffen, dass sich $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$ in beliebiger Dimension d wie $(\log N)^{d-1}/N^2$ verhält. Deshalb präsentieren wir numerische Simulationen, die uns zu Vermutung 4.1 veranlassen:

Vermutung. *Für den Erwartungswert der optimal gewichteten L_2-Diskrepanz in beliebiger Dimension d gilt*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{(\log N)^{d-1}}{N^2}.$$

Introduction

While todays research in discrepancy theory affects different branches of mathematics like combinatorics or probability theory, it originates in the theory of uniform distribution with a fundamental work of Weyl in 1916 ([53]) and a conjecture of van der Corput in 1935 ([11, 12]):

Conjecture. *If* (s_i) *is an infinite sequence of real numbers between 0 and 1, then - corresponding to any arbitrarily large* κ *- there exists a positive integer* n *and two subintervals of* $(0,1)$ *of equal length, such that the number of* s_i $(i = 1, \dots, n)$ *that lie in one of the subintervals differs from the number of such* s_ν *that lie in the other subinterval by more then* κ*.*

This conjecture claims that there exists no perfectly uniformly distributed sequence. In 1945, it was proved by van Aardenne-Ehrenfest ([3]).

In 1954, Roth published a milestone in discrepancy theory ([43]). He used a different notion for the problem above which included the discrepancy function. In this work, we define the discrepancy function by

$$D(B, \mathcal{P}) = \frac{1}{N} \sum_{j=1}^{N} \chi_B(t_j) - \mathrm{vol}(B)$$

for some $B = [0, x] \subset [0,1]^d$ and some $\mathcal{P} = \{t_1, \dots, t_N\} \subset [0,1]^d$. Roth showed

$$\int_{[0,1]^2} (N D(B, \mathcal{P}))^2 \, \mathrm{d}x > c \log N$$

for any point set $\mathcal{P} \subset [0,1]^2$. This result was the starting point for intensive research in discrepancy theory. For a good historical overview of discrepancy theory, from the beginning to the 1980s, we refer to [4].

Usually, one is interested in the norm of the discrepancy function in some function spaces. In this work we investigate the classical case of L_p norms for $0 < p \leq \infty$. This leads to the definition of the L_p-star discrepancy

$$D_p^*(\mathcal{P}) = \left(\int_{[0,1]^d} |D(B, \mathcal{P})|^p \, \mathrm{d}x \right)^{1/p}$$

for $0 < p < \infty$, and

$$D^*_\infty(\mathcal{P}) = \sup_{x\in[0,1]^d} |D(B,\mathcal{P})|$$

for $p = \infty$.

The main goal of this work is to investigate expected discrepancies: We choose a random point set $\mathcal{P}$ of independent an uniformly distributed points in $[0,1]^d$ and compute the expectation of L_p-discrepancies.

In Chapter 1 we fix relevant notation, summarize definitions and results of probability theory and give a short introduction in discrepancy theory. Subsequently, this work is split in three parts with - in large parts - independent Chapters 2, 3 and 4.

In Chapter 2 we analyse expected L_p-discrepancies for $0 < p < \infty$. To this end, we generalize the notion of discrepancy to the L_p-B-discrepancy, which means that we allow general sets B instead of anchored boxes $[0,x]$ (see Definition 1.18). We define the average L_p-discrepancy as

$$\mathrm{av}^*_p(N,d) = \left(\mathbb{E}\left(D^*_p(\mathcal{P})^p\right)\right)^{1/p}.$$

The expectation is taken over sets $\mathcal{P}$ of N independent and uniformly distributed points $t_1,\dots,t_N \in [0,1]^d$. We further define the average L_p-B-discrepancy as

$$\mathrm{av}^B_p(N,d) = \left(\mathbb{E}\left(D^B_p(\mathcal{P})^p\right)\right)^{1/p}.$$

The first definition was introduced by Heinrich, Novak, Wasilkowski and Woźniakowski in [30], who proved the equality

$$\mathrm{av}^*_p(N,d)^p = \sum_{r=p/2}^{p-1} C(r,p,d)N^{-r}$$

for even p and with known constants $C(r,p,d)$. For arbitrary p, in [50] Steinerberger published a limit relation in N, but with a gap in his proof. The main result of Chapter 2 is a generalization of Steinerberger's result for the average L_p-B-discrepancy, formulated in Theorem 2.2:

Theorem. *Let $p > 0$ and $d \in \mathbb{N}$, further let (Ω_d,μ_d) be a probability space and $\{B(x) : x \in \Omega_d\} \subset 2^{[0,1]^d}$ the allowed sets. Then*

$$\lim_{N\to\infty} N^{p/2}\mathrm{av}^B_p(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int_{\Omega_d}\left[\lambda^d(B(x))\big(1-\lambda^d(B(x))\big)\right]^{p/2}\mathrm{d}\mu_d(x).$$

This result was published in [35].

In Chapter 3 we investigate expected star discrepancies which is the case $p = \infty$. The best known upper bound for the minimal star discrepancy was shown by Heinrich, Novak, Wasilkowski and Woźniakowski in [30]. They proved

$$D^*(N,d) = \inf_{\substack{\mathcal{P}\subset[0,1]^d \\ \#\mathcal{P}=N}} D^*_\infty(\mathcal{P}) \leq c\,\sqrt{\frac{d}{N}}\,.$$

No matching lower bounds are known for $D^*(N,d)$. The best known lower bound is due to Hinrichs ([32]) who showed

$$D^*(N,d) \geq \min\left\{\varepsilon_0, c\,\frac{d}{N}\right\}$$

with constants $c, \varepsilon_0 > 0$. In [18] Doerr asked for random point sets instead of the optimal ones and showed the existence of a constant $K > 0$ such that

$$\mathbb{E}D^*_\infty(\mathcal{P}) \geq K\sqrt{\frac{d}{N}}\,.$$

Interestingly, considering an a-weighted discrepancy, it holds for $d = N = 1$ that

$$\inf_a \mathbb{E}D^*_\infty(a,\mathcal{P}) < \mathbb{E}D^*_\infty(\mathcal{P}),$$

which motivates us to compute a lower bound in the weighted case. This yields Theorem 3.3, the main result of Chapter 3:

Theorem. *There is an absolute constant $K > 0$ such that the following is true: Let $N, d \in \mathbb{N}$ with $d \leq N$. Let $\mathcal{P}$ be a set of N points chosen independently and uniformly at random from $[0,1]^d$. Let further $a = (a_j)_{j=1}^N$ be a sequence of weights. Then the expected weighted star discrepancy satisfies*

$$\mathbb{E}D^*_\infty(a,\mathcal{P}) \geq K \cdot \sqrt{\frac{d}{N}}\,.$$

Finally, in Section 3.3 we ask for the expectation of the weighted star discrepancy not with arbitrary, fixed weights but with the optimal ones. This means that we choose the weights for a fixed point set in such a way that the discrepancy gets minimal. For dimension $d = 1$ we show Theorem 3.6:

Theorem. *For the expectation of the optimally weighted star discrepancy in dimen-*

sion $d = 1$ we have the asymptotic behavior

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \asymp \frac{\log N}{N}.$$

In Chapter 4 we return to the case of the L_2-discrepancy. From the famous result of Roth ([43]) we know that that

$$D_2^*(\mathcal{P}) \geq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

holds for any $\mathcal{P} \subset [0,1]^d$ with N points. In 2002, Chen and Skriganov constructed an explicit point set with

$$D_2^*(\mathcal{P}) \leq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

for the first time ([10]). Up to that point, only probabilistic constructions were known ([22, 46]). From the work of Heinrich, Novak, Wasilkowski and Woźniakowski ([30]) we know

$$\mathbb{E} D_2^*(\mathcal{P})^2 = C_d \frac{1}{N}.$$

This means that random point sets yield worse order in N for the L_2-discrepancy than the optimal point sets. Inspired by the result for the optimally weighted star discrepancy in Chapter 3 we compute the expectation of the optimally weighted L_2-discrepancy and hope for better asymptotic behavior in N. In Theorem 4.4 we show:

Theorem. *For the expectation of the optimally weighted L_2-discrepancy in dimension $d = 1$ the following relation holds*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \inf_{\substack{\mathcal{P} \subset [0,1] \\ \#\mathcal{P} = N}} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{1}{N^2}.$$

This theorem gives hope that the asymptotic behavior of $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$ in arbitrary dimension d is $(\log N)^{d-1}/N^2$. Hence, we present numerical simulations which lead to Conjecture 4.1:

Conjecture. *For the expectation of the optimally weighted L_2-discrepancy in arbitrary dimension d the following relation holds*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{(\log N)^{d-1}}{N^2}.$$

1 Preliminaries

1.1 Basic notation

By $d \in \mathbb{N}$ we will denote the dimension, $[0,1]^d$ is the d-dimensional unit cube. For fixed $x \in [0,1]^d$ we denote by $B = B(x)$ the d-dimensional box with lower left corner 0 and upper right corner x:

$$B(x) = [0,x] = \left\{ y \in [0,1]^d : 0 \leq y_i \leq x_i \text{ for all } i = 1, \dots, d \right\}.$$

With χ_B we denote the characteristic function of the set B:

$$\chi_B(t) = \begin{cases} 1, & \text{if } t \in B, \\ 0, & \text{otherwise.} \end{cases}$$

By $N \in \mathbb{N}$ we will denote the cardinality of a point set; an arbitrary point set in the d-dimensional unit cube will be denoted by $\mathcal{P} = \{t_1, \dots, t_N\} \subset [0,1]^d$. We also write $|\mathcal{P}|$ for the cardinality of a point set.

We denote by $[N] = \{1, \dots, N\}$ the set of the first N natural numbers.

1.2 Probability theory

In this section we want to recall some basic definitions and results of probability theory. For detailed information we refer to [13], [36] and [38].

1.2.1 Probabilistic basics

For the following definitions and results we refer to [36].

Definition 1.1. Let $\Omega \neq \emptyset$. A subset $\mathcal{A} \subset 2^\Omega$ is called a ***σ-algebra*** if the following properties are fulfilled:

(i) $\Omega \in \mathcal{A}$.

(ii) $\mathcal{A}$ is closed under complementation, that is: $A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A}$.

(iii) $\mathcal{A}$ is closed under countable unions, that is: $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ is called a ***measurable space***.

Example 1.1. We denote by $\mathcal{B}(\mathbb{R}^d)$ the ***Borel σ-algebra*** over $\mathbb{R}^d$ which is defined as the smallest σ-algebra which contains all open sets in $\mathbb{R}^d$. The elements of $\mathcal{B}(\mathbb{R}^d)$ are called Borel sets.
More general, one can define Borel σ-algebras over all topological spaces.

Definition 1.2. Let $\mathcal{A} \subset 2^\Omega$ be a σ-algebra and $\mu : \mathcal{A} \to [0, \infty)$. The function μ is called ***σ-additive*** if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for any sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint sets in $\mathcal{A}$.

Definition 1.3. Let $\mathcal{A} \subset 2^\Omega$ be a σ-algebra. The function $\mu : \mathcal{A} \to [0, \infty)$ is called a ***measure*** if μ is σ-additive and $\mu(\emptyset) = 0$. The triple $(\Omega, \mathcal{A}, \mu)$ is then called a ***measure space***.

If additionally $\mu(\Omega) = 1$ holds, then μ is called a ***probability measure***. The triple $(\Omega, \mathcal{A}, \mu)$ is then called a ***probability space*** and the sets $A \in \mathcal{A}$ are called ***events***. In this case we often write $\mu = \mathbb{P}$.

Definition 1.4. Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces.

(i) A mapping $X : \Omega \to \Omega'$ is called ***measurable*** if

$$X^{-1}(A') \in \mathcal{A} \text{ for all } A' \in \mathcal{A}'.$$

(ii) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X : \Omega \to \Omega'$ measurable. Then X is called an ***Ω'-valued random variable***. If $(\Omega', \mathcal{A}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then X is called a real-valued random variable or simply ***random variable***.

(iii) Let X be a random variable. The probability measure $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ is called ***probability distribution of X***. We write $X \sim \mathbb{P}_X$.

(iv) Let X be a random variable. The mapping $F_X : x \mapsto \mathbb{P}(X \leq x)$ is called the ***cumulative distribution function of X***.

Now, we summarize some facts about $L_p(\mu)$ spaces and expectations.

Definition 1.5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

(i) Let $f : \Omega \to \mathbb{R} \cup \{+\infty, -\infty\}$ measurable. For $p \in [1, \infty)$ we define the norm

$$\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p},$$

and for $p = \infty$

$$\|f\|_\infty = \inf\{K \geq 0 : \mu(\{|f| > K\}) = 0\}.$$

For $p \in [1, \infty]$ we define the vector space

$$L_p(\mu) = \Big\{ f : \Omega \to \mathbb{R} \cup \{+\infty, -\infty\} : f \text{ measurable and } \|f\|_p < \infty \Big\}.$$

(ii) Let now $\mu = \mathbb{P}$ be a probability measure and X be a random variable with $X \in L_1(\mathbb{P})$. Then we define by

$$\mathbb{E}(X) = \int X \, \mathrm{d}\mathbb{P}$$

the ***expectation of X***.

Lemma 1.1 (Law of the unconscious statistician)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X a random variable. Let further $g : \mathbb{R} \to \mathbb{R}$ be measurable. Then it holds*

$$\mathbb{E}\left(g\left(X\right)\right) = \int_{\mathbb{R}} g(x) \, \mathrm{d}F_X(x).$$

An often used special case of this lemma is the identity

$$\mathbb{E}\left(|X|^\alpha\right) = \alpha \int_0^\infty t^{\alpha-1} \mathbb{P}\left(|X| > t\right) \, \mathrm{d}t \tag{1.2.1}$$

for $\alpha > 0$.

Definition 1.6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X a random variable on it. The mapping $\varphi_X : \mathbb{R} \to \mathbb{C}$, defined as

$$\varphi_X(t) = \int_\Omega e^{itX} \, \mathrm{d}\mathbb{P}$$

is called the ***characteristic function of X***.

Finally, we mention two famous inequalities.

Theorem 1.1 (Markov's Inequality)**.** *Let X be a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $f : [0, \infty) \to [0, \infty)$ monotonically increasing. Then, for any $\varepsilon > 0$ with $f(\varepsilon) > 0$, the inequality*

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}\Big(f(|X|)\Big)}{f(\varepsilon)}$$

holds.

Theorem 1.2 (Hölder's Inequality)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let further $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L_p(\mu), g \in L_q(\mu)$. Then $f \cdot g \in L_1(\mu)$ and*

$$\|f \cdot g\|_1 \leq \|f\|_p \, \|g\|_q \, .$$

If μ is a probability measure and the functions $f = X$ and $g = Y$ are random variables then the integrals can be written as expectations

$$\mathbb{E}\,|XY| \leq \Big(\mathbb{E}\,|X|^p\Big)^{1/p} \Big(\mathbb{E}\,|Y|^q\Big)^{1/q}.$$

1.2.2 Some probability distributions and their properties

In this section we gather some knowledge about two famous probability distributions, namely the binomial distribution and the uniform distribution on $[0, 1]$.

Binomial distribution

Definition 1.7. Let $n \in \mathbb{N}$ and $p \in [0, 1]$. Let further $X : \Omega \to \{0, 1, \ldots, n\}$ with

$$\mathbb{P}\,(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for all $k \in \{0, 1, \ldots, n\}$. Then $\mathbb{P}_X = B_{n,p}$ is called the ***binomial distribution*** with parameters n and p.

The median m of the binomial distribution satisfies $\lfloor np \rfloor \leq m \leq \lceil np \rceil$.

We need the following lemmata about the binomial distribution.

Lemma 1.2. *For fixed n and $k \in \mathbb{N}$ with $0 \leq k \leq n$, the binomial distribution $B_{n,p}(k)$ is maximal for $p = k/n$.*

Proof. We define

$$f_{n,k}(p) = B_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

For $k = 0$ and $k = n$ Lemma 1.2 is obviously true. Let now $1 \leq k < n$. We differentiate $f_{n,k}$ and get

$$f'_{n,k}(p) = \binom{n}{k} p^{k-1} (1-p)^{n-k-1} \left[k(1-p) - (n-k)p\right].$$

Hence, we have

$$f'_{n,k}(p) = 0 \Leftrightarrow k(1-p) - (n-k)p = 0 \Leftrightarrow p = k/n.$$

Because of $f_{n,k}(0) = 0, f_{n,k}(1) = 0$, $f_{n,k}(k/n) > 0$ and the continuity of $f_{n,k}$, the point $p = k/n$ is a global and local maximum. □

Lemma 1.3. *Let $n \in \mathbb{N}$. For $k \leq n/2$ the binomial distribution $B_{n,\frac{k}{n}}(k)$ is monotonically decreasing as a function of k, for $k \geq n/2$ it is monotonically increasing.*

Proof. We first show that $B_{n,\frac{k}{n}}(k)$ is symmetric about $n/2$:

$$\begin{aligned} B_{n,\frac{n-k}{n}}(n-k) &= \binom{n}{n-k}\left(\frac{n-k}{n}\right)^{n-k}\left(1-\frac{n-k}{n}\right)^{n-(n-k)} \\ &= \binom{n}{k}\left(1-\frac{k}{n}\right)^{n-k}\left(\frac{k}{n}\right)^{k} \\ &= B_{n,\frac{k}{n}}(k). \end{aligned}$$

Hence, it suffices to show that for $k = 0, \ldots, \lfloor n/2 \rfloor$ the function $B_{n,\frac{k}{n}}(k)$ is monotonically decreasing.

For $n = 1$ we have $k = 0$ and consequently $B_{n,\frac{k}{n}}(k) = 1$, which is monotonically decreasing. If $k = 0$ then again $B_{n,\frac{k}{n}}(k) = 1$.

Let now $n \geq 2$ and $1 \leq k \leq n/2$. We define

$$\begin{aligned} f(n,k) &= \frac{B_{n,\frac{k-1}{n}}(k-1)}{B_{n,\frac{k}{n}}(k)} \\ &= \frac{\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k-1}\left(1-\frac{k-1}{n}\right)^{n-(k-1)}}{\binom{n}{k}\left(\frac{k}{n}\right)^{k}\left(1-\frac{k}{n}\right)^{n-k}} \\ &= \frac{n!k!(n-k)!(n-k+1)^{n-k+1}(k-1)^{k-1}n^{n-k}n^{k}}{n!(k-1)!(n-k+1)!n^{n-k+1}n^{k-1}(n-k)^{n-k}k^{k}} \\ &= \frac{k(n-k+1)^{n-k+1}(k-1)^{k-1}}{(n-k+1)(n-k)^{n-k}k^{k}} \\ &= \frac{(n-k+1)^{n-k}}{(n-k)^{n-k}} \cdot \left(\frac{k-1}{k}\right)^{k-1}. \end{aligned}$$

We show $f(n,k) \geq 1$ for all $n \in \mathbb{N}$ and $1 \leq k \leq n/2$. To this end we factorize

$$f(n,k) = h(n-k) \cdot g(k)$$

with

$$h(x) = \left(\frac{x+1}{x}\right)^{x} \text{ and } g(x) = \left(\frac{x-1}{x}\right)^{x-1}.$$

The function h is monotonically increasing on $[1,\infty)$. This is true because

$$h'(x) = x\left(\frac{x+1}{x}\right)^{x-1}\left(-\frac{1}{x^2}\right) = 0 \Leftrightarrow x+1 = 0 \Leftrightarrow x = -1.$$

Therefore h has one extremal point which is not in the interval $[1,\infty)$. Hence, h is monotonically increasing on $[1,\infty)$ because h is continuous on this interval, $h(1) = 2$ and $\lim_{x\to\infty} h(x) = e$. This gives for all $k \leq n/2$ the estimate

$$f(n,k) = h(n-k)g(k) \geq h(2k-k)g(k) = h(k)g(k) = \tilde{f}(k).$$

The proof is complete, if we can show $\tilde{f}(k) \geq 1$ for all $1 \leq k \leq n/2$. We have

$$\begin{aligned}\tilde{f}'(x) &= h'(x)g(x) + h(x)g'(x)\\ &= \frac{(x+1)^{x-1}}{-x^x}\left(\frac{x-1}{x}\right)^{x-1} + \left(\frac{x+1}{x}\right)^{x}\frac{(x-1)^{x-1}}{x^x}\end{aligned}$$

and therefore

$$\tilde{f}'(x) = 0 \Leftrightarrow \frac{(x+1)^{x-1}(x-1)^{x-1}}{x^{2x-1}} = \frac{(x+1)^{x}(x-1)^{x-1}}{x^{2x}} \Leftrightarrow 1 = \frac{x+1}{x} = 1 + \frac{1}{x},$$

which can never be true. Hence $\tilde{f}$ has no extremal point. Because $\tilde{f}$ is further continuous on $[1,\infty)$, $\tilde{f}(1) = 2$ and $\lim_{n\to\infty} \tilde{f}(x) = 1$, we always have $\tilde{f}(k) \geq 1$ for all $k \geq 1$. □

Further, we present a lemma from Doerr (see [18]).

Lemma 1.4. *(Doerr) Let X be a binomial distributed random variable with parameters $n \in \mathbb{N}$ and $p = 1/2$. Then it holds*

$$\mathbb{P}\left(X \leq \frac{n}{2} - \frac{1}{2}\sqrt{\frac{n}{2}}\right) \geq \frac{1}{8}.$$

Because the binomial distribution for $p = 1/2$ is symmetric about the expectation $n/2$ it also holds

$$\mathbb{P}\left(X \geq \frac{n}{2} + \frac{1}{2}\sqrt{\frac{n}{2}}\right) \geq \frac{1}{8}.$$

These three lemmata together finally give the following result.

Lemma 1.5. *(Doerr) Let $n \geq 16$ and $1/n \leq p \leq 1/4$. Let X be a binomial distributed random variable with parameters n and p. Then it holds*

$$\mathbb{P}\left(X \leq pn - \frac{\sqrt{pn}}{2}\right) \geq 3/160$$

and

$$\mathbb{P}\left(X \geq pn + \frac{\sqrt{pn}}{2}\right) \geq 3/160.$$

The first inequality was proved by Doerr in [18]. We prove here the second inequality to show the idea of Doerr's proof.

Proof. Let Y be a random subset of $[n]$ with

$$\mathbb{P}(i \in Y) = 2p$$

for all $i \in [n]$. Hence, the random variable $y = |Y|$ is binomial distributed with parameters n and $2p$. Let further Z be a random subset of Y with

$$\mathbb{P}(i \in Z) = 1/2$$

for all $i \in Y$. Hence, the random variable $z = |Z|$ is binomial distributed with parameters n and p.

By m_y we denote the median of y, for which

$$\lfloor 2pn \rfloor \leq m_y \leq \lceil 2pn \rceil .$$

This gives

$$\begin{aligned}
\mathbb{P}\left(y \geq \lceil 2pn \rceil\right) &\geq \mathbb{P}(y \geq \lfloor 2pn \rfloor) - \mathbb{P}(y = \lfloor 2pn \rfloor) \\
&\geq \mathbb{P}(y \geq m_y) - \mathbb{P}(y = \lfloor 2pn \rfloor) \\
&\geq 1/2 - \mathbb{P}(y = \lfloor 2pn \rfloor). \qquad (1.2.2)
\end{aligned}$$

Because the binomial distribution $B_{n,p}(k)$ is maximal for $p = k/n$ (see Lemma 1.2), we get

$$\begin{aligned}
\mathbb{P}(y = \lfloor 2pn \rfloor) &= B_{n,2p}\left(\lfloor 2pn \rfloor\right) \\
&\leq B_{n,\frac{\lfloor 2pn \rfloor}{n}}\left(\lfloor 2pn \rfloor\right).
\end{aligned}$$

Further, the function

$$B_{n,k/n}(k) = \binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}$$

is decreasing in k for $k = 0, \dots, \lfloor n/2 \rfloor$ and increasing in k for $k = \lceil n/2, \dots, n \rceil$ (see Lemma 1.3). This gives

$$\begin{aligned}
\mathbb{P}(y = \lfloor 2pn \rfloor) &\leq B_{n,\frac{\lfloor 2pn \rfloor}{n}}\left(\lfloor 2pn \rfloor\right) \\
&\leq B_{n,\frac{2}{n}}(2) \\
&= \binom{n}{2}\left(\frac{2}{n}\right)^2\left(1-\frac{2}{n}\right)^{n-2} \\
&= \frac{n!}{2(n-2)!}\frac{4}{n^2}\left(1-\frac{2}{n}\right)^{n-2} \\
&= \frac{2n(n-1)}{n^2}\left(1-\frac{2}{n}\right)^{n-2} \\
&\leq 2\left(1-\frac{2}{n}\right)^{n-2} \\
&\leq 2e^{-\frac{2(n-2)}{n}} \\
&\leq 2e^{\frac{1}{4}-2} \\
&\leq 0.35.
\end{aligned}$$

Together with (1.2.2) we get

$$\mathbb{P}\left(y \geq \lceil 2pn \rceil\right) \geq 0.15. \tag{1.2.3}$$

Let

$$f(x) = \frac{x}{2} + \frac{\sqrt{x/2}}{2}.$$

This function is monotonically increasing for all $x \geq 0$. In particular we have for all $\lceil 2pn \rceil \leq y \leq n$

$$f\left(\lceil 2pn \rceil\right) \leq f(y) \leq f(n).$$

Now, we can apply Lemma 1.4 and get for fixed $y \in [\lceil 2pn \rceil, n]$ the estimate

$$\mathbb{P}\left(z \geq f\left(\lceil 2pn \rceil\right)\right) \geq \mathbb{P}\left(z \geq f(y)\right) \geq \mathbb{P}\left(z \geq f(n)\right) \geq 1/8. \tag{1.2.4}$$

Combining (1.2.3) and (1.2.4) we get

$$\begin{aligned}
\mathbb{P}\left(z \geq pn + \frac{\sqrt{pn}}{2}\right) &= \mathbb{P}\left(z \geq f\left(2pn\right)\right) \\
&\geq \mathbb{P}\left(z \geq f\left(\lceil 2pn \rceil\right)\right) \\
&= \sum_{i=0}^{n} \mathbb{P}(y = i)\mathbb{P}\left(z \geq f\left(\lceil 2pn \rceil\right) \mid y = i\right) \\
&\overset{(1.2.3)}{\geq} \sum_{i=\lceil 2pn \rceil}^{n} \mathbb{P}(y = i) \cdot 1/8 \\
&= 1/8 \cdot \mathbb{P}\left(y \geq \lceil 2pn \rceil\right) \\
&\overset{(1.2.4)}{\geq} 3/160.
\end{aligned}$$

□

Uniform distribution on $[0,1]$ and random division of $[0,1]$

In this section we want to briefly discuss the random division of the unit interval $[0,1]$. For further information we refer to [13].

Definition 1.8. Let $X : \Omega \to [0,1]$ be a random variable and let $I = [a,b] \subset [0,1]$. Then X is called ***uniformly distributed*** on $[0,1]$ if

$$\mathbb{P}\left(X \in I\right) = b - a.$$

Let $X_1, \ldots, X_n$ be independent random variables, uniformly distributed on $[0,1]$. We are interested in the behavior of the random variables $Y_0, Y_1, \ldots, Y_n$, which are the lengths of the $n+1$ segments into which the unit interval is divided by the X_i. If (X_i^*) is a rearrangement of the X_i with $0 = X_0^* \leq X_1^* \leq \ldots X_n^* \leq X_{n+1}^* = 1$, the Y_i are defined as

$$Y_i = X_{i+1}^* - X_i^*,$$

for $i = 0, \ldots, n$.

The following two theorems were shown by Darling in [13].

Theorem 1.3. *Let $h : [0,\infty) \to \mathbb{R}$ be an integrable function with*

$$\int_0^\infty h(r)\,\mathrm{d}r < \infty.$$

and

$$W_n = \sum_{i=0}^{n} h(Y_i).$$

Then the following equality is true

$$\mathbb{E}W_n = n(n+1)\int_0^1 (1-r)^{n-1} h(r)\,\mathrm{d}r. \tag{1.2.5}$$

Theorem 1.4. *Let $V_n = \max\{Y_0, \dots, Y_n\}$ and $\alpha \in \mathbb{R}$. Then it holds*

$$\lim_{n\to\infty} \mathbb{P}\left(V_n < \frac{\log(N+1)+\alpha}{N+1}\right) = e^{-e^{-\alpha}}.$$

1.2.3 Convergence in probability theory

In this section we define several types of convergence in probability theory and present some results. If not otherwise stated, we refer to [36].

Theorem 1.5 (Fatou's Lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f \in L_1(\mu)$ and (f_n) measurable with $f_n \geq f$ μ-almost everywhere for all $n \in \mathbb{N}$. Then it holds*

$$\int_\Omega \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu.$$

Definition 1.9. Let E be a metric space and $\mu, (\mu_n)$ finite measures on $(E, \mathcal{B}(E))$. The measures μ_n ***converge weakly*** to μ if

$$\lim_{n\to\infty} \int_E f \,\mathrm{d}\mu_n = \int_E f \,\mathrm{d}\mu$$

for all bounded and continuous functions f on E. We write

$$\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n.$$

Definition 1.10. Let $X, (X_n)$ be random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The sequence (X_n) is said to ***converge in distribution*** to the random variable X if $\mathbb{P}_X = \operatorname*{w-lim}_{n\to\infty} \mathbb{P}_{X_n}$. We write

$$X_n \xrightarrow{\mathcal{D}} X.$$

Theorem 1.6 (Lévy-Cramér Continuity Theorem)**.** *Let $(\mathbb{P}_n)$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with characteristic functions (φ_n). Then*

$$\mathbb{P} = \operatorname*{w-lim}_{n\to\infty} \mathbb{P}_n \Leftrightarrow \lim_{n\to\infty} \varphi_n = \varphi.$$

Theorem 1.7 (Portemanteau Theorem)**.** *Let E be a metric space and let $\mu, (\mu_n)$ be finite measures on $(E, \mathcal{B}(E))$ with $\mu_n(E) \leq 1$ and $\mu(E) \leq 1$. Then the following assertions are equivalent*

(i) $\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n$,

(ii) $\lim_{n\to\infty} \int_E f \,\mathrm{d}\mu_n = \int_E f \,\mathrm{d}\mu$ *for all bounded and Lipschitz functions* f,

(iii) $\lim_{n\to\infty} \int_E f \,\mathrm{d}\mu_n = \int_E f \,\mathrm{d}\mu$ *for all bounded and measurable functions* f *with* $\mu(U_f) = 0$; U_f *is the set of all points of discontinuity of* f,

(iv) $\liminf_{n\to\infty} \mu_n(E) \geq \mu(E)$ *and* $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ *for all closed sets* $F \subset E$,

(v) $\limsup_{n\to\infty} \mu_n(E) \leq \mu(E)$ *and* $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ *for all open sets* $G \subset E$,

(vi) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ *for all measurable sets* A *with* $\mu(\partial A) = 0$.

Theorem 1.8 (Continuous Mapping Theorem)**.** *Let E_1 and E_2 be metric spaces and $f : E_1 \to E_2$ be measurable. Let further U_f be the set of all points of discontinuity of f.*

(i) *Let $\mu, (\mu_n)$ be finite measures on $(E_1, \mathcal{B}(E_1))$ with $\mu_n(E_1) \leq 1$ and $\mu(E_1) \leq 1$. Let further $\mu(U_f) = 0$ and $\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n$. Then*

$$\mu \circ f^{-1} = \operatorname*{w-lim}_{n\to\infty} \mu_n \circ f^{-1}.$$

(ii) *Let $X, (X_n)$ be E_1-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{P}(X \in U_f) = 0$ and $X_n \xrightarrow{\mathcal{D}} X$. Then*

$$f(X_n) \xrightarrow{\mathcal{D}} f(X).$$

Lemma 1.6. *Let $X, (X_n)$ be random variables with $X_n \xrightarrow{\mathcal{D}} X$. Then it holds*

(i) $\mathbb{E}\left(|X|\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(|X_n|\right)$.

(ii) *If $p > 0$ and $\sup_{n\in\mathbb{N}} \mathbb{E}\left(|X_n|^r\right) < \infty$ for one $r > p$, then* $\mathbb{E}\left(|X|^p\right) = \lim_{n\to\infty} \mathbb{E}\left(|X_n|^p\right)$.

Proof. (i) Let $X_n \xrightarrow{\mathcal{D}} X$. Because of Theorem 1.8 (ii) we have

$$|X_n| \xrightarrow{\mathcal{D}} |X|\,.$$

Now, we apply Theorem 1.7 (v) and get

$$\liminf_{n\to\infty} \mathbb{P}_{|X_n|}(G) \geq \mathbb{P}_{|X|}(G) \tag{1.2.6}$$

for all open sets G. Combining (1.2.1) and (1.2.6) yields

$$\begin{aligned}\mathbb{E}(|X|) &= \int_0^\infty \mathbb{P}(|X| > t)\,\mathrm{d}t \\ &= \int_0^\infty \mathbb{P}_{|X|}\Big((t,\infty)\Big)\,\mathrm{d}t \\ &\le \int_0^\infty \liminf_{n\to\infty} \mathbb{P}_{|X_n|}\Big((t,\infty)\Big)\,\mathrm{d}t.\end{aligned}$$

Now, we apply Theorem 1.5 and get

$$\begin{aligned}\mathbb{E}(|X|) &\le \liminf_{n\to\infty} \int_0^\infty \mathbb{P}_{|X_n|}\Big((t,\infty)\Big)\,\mathrm{d}t \\ &\le \liminf_{n\to\infty} \int_0^\infty \mathbb{P}(|X_n| > t)\,\mathrm{d}t \\ &= \liminf_{n\to\infty} \mathbb{E}(|X_n|).\end{aligned}$$

(ii) Let now $X_n \xrightarrow{\mathcal{D}} X$, $p > 0$ and $\sup_{n\in\mathbb{N}} \mathbb{E}(|X_n|^r) < \infty$ for one $r > p$.

Because of Theorem 1.8 (ii) we have

$$X_n^p \xrightarrow{\mathcal{D}} X^p,$$

applying part (i) of this lemma we get

$$\mathbb{E}(|X|^p) \le \liminf_{n\to\infty} \mathbb{E}(|X_n|^p). \tag{1.2.7}$$

We now set

$$M = \sup_{n\in\mathbb{N}} \mathbb{E}(|X_n|^r) < \infty.$$

Let $N \in \mathbb{N}$ be arbitrary. We choose $\varepsilon = M^{1/r}N$ and $f(x) = x^r$ and apply Markov's inequality (Theorem 1.1) for X_n. This yields

$$\mathbb{P}(|X_n| > M^{1/r}N) \le \frac{\mathbb{E}(|X_n|^r)}{MN^r} = \frac{\mathbb{E}(|X_n|^r)}{\sup_{n\in\mathbb{N}} \mathbb{E}(|X_n|)^r N^r} \le \frac{1}{N^r} \tag{1.2.8}$$

for all $n, N \in \mathbb{N}$.

We now separate (1.2.7) to

$$\mathbb{E}(|X|^p) \le \liminf_{n\to\infty} \Big[\mathbb{E}(|X_n|^p \mathbb{1}(|X_n| < \varepsilon)) + \mathbb{E}(|X_n|^p \mathbb{1}(|X_n| \ge \varepsilon))\Big]. \tag{1.2.9}$$

Because $f(x) = |x|^p \mathbb{1}(|x| < \varepsilon)$ is bounded and measurable we can apply Theorem 1.7 (iii) and get

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left(|X_n|^p \mathbb{1}\left(|X_n| < \varepsilon\right)\right) &= \lim_{n\to\infty} \int\limits_{\Omega} f \,\mathrm{d}\mathbb{P}_{X_n} \\ &= \int\limits_{\Omega} f \,\mathrm{d}\mathbb{P}_X \\ &= \mathbb{E}\left(|X|^p \mathbb{1}\left(|X| < \varepsilon\right)\right) \\ &\leq \mathbb{E}\left(|X|^p\right). \qquad (1.2.10)\end{aligned}$$

On the second summand of (1.2.9) we apply Hölder's Theorem (Theorem 1.2) and get

$$\begin{aligned}\mathbb{E}\left(|X_n|^p \mathbb{1}\left(|X_n| \geq \varepsilon\right)\right) &\leq \left(\mathbb{E}\,|X_n|^r\right)^{p/r} \left(\mathbb{E}\left(\mathbb{1}\left(|X_n| \geq \varepsilon\right)\right)^{r/(r-p)}\right)^{(r-p)/r} \\ &\leq M^{p/r}\mathbb{P}\left(|X_n| \geq M^{1/r}N\right)^{(r-p)/r}.\end{aligned}$$

Together with (1.2.8) this yields

$$\mathbb{E}\left(|X_n|^p \mathbb{1}\left(|X_n| \geq \varepsilon\right)\right) \leq M^{p/r}\left(\frac{1}{N^r}\right)^{(r-p)/r} = \frac{M^{p/r}}{N^{r-p}}. \qquad (1.2.11)$$

Combining now (1.2.7), (1.2.9), (1.2.10) and (1.2.11) we get for all $N \in \mathbb{N}$

$$\mathbb{E}\left(|X|^p\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(|X_n|^p\right) \leq \limsup_{n\to\infty}\left[\mathbb{E}\left(|X|^p\right) + \frac{M^{p/r}}{N^{r-p}}\right] = \mathbb{E}\left(|X|^p\right) + \frac{M^{p/r}}{N^{r-p}}.$$

Letting now $N \to \infty$ we get

$$\mathbb{E}\left(|X|^p\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(|X_n|^p\right) \leq \limsup_{n\to\infty} \mathbb{E}\left(|X_n|^p\right) \leq \mathbb{E}\left(|X|^p\right)$$

which yields

$$\mathbb{E}\left(|X|^p\right) = \lim_{n\to\infty} \mathbb{E}\left(|X_n|^p\right).$$

□

Theorem 1.9 (Dominated Convergence Theorem)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, f measurable on $(\Omega, \mathcal{A}, \mu)$ and $(f_n) \in L_1(\mu)$ with $\lim_{n\to\infty} f_n = f$ pointwise. Let further $g \in L_1(\mu)$ with $g \geq 0$ and $|f_n| \leq g$ μ-almost everywhere for all $n \in \mathbb{N}$. Then $f \in L_1(\mu)$ and*

$$\lim_{n\to\infty} \int_{\Omega} |f_n - f| \,\mathrm{d}\mu = 0$$

which especially implies

$$\lim_{n\to\infty}\int_\Omega f_n\,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu.$$

Theorem 1.10 (Central limit Theorem). *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with expectation $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. For $n \in \mathbb{N}$ let*

$$S_n = \frac{1}{\sqrt{\sigma^2 n}}\sum_{i=1}^{n}(X_i - \mu).$$

Then it holds

$$S_n \xrightarrow{\mathcal{D}} Y,$$

with $Y \sim \mathcal{N}(0,1)$.

Theorem 1.11 (Berry-Esseen Theorem). *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with expectation $\mu = 0$, variance $\sigma^2 > 0$ and third absolute moment $\gamma = \mathbb{E}\left(|X_i|^3\right) < \infty$. For $n \in \mathbb{N}$ let*

$$S_n = \frac{1}{\sqrt{\sigma^2 n}}\sum_{i=1}^{n} X_i.$$

Then it holds for all $n \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(S_n \le x\right) - F_Y(x)\right| \le \frac{0.8\gamma}{\sigma^3\sqrt{n}}$$

with $Y \sim \mathcal{N}(0,1)$.

1.2.4 Banach space valued random variables

In this section we discuss random variables whose target spaces are Banach spaces. We just present some facts about such random variables. For detailed information we refer to [38].

Definition 1.11. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and B a Banach space. A ***simple function*** $s : \Omega \to B$ is any finite sum of the form

$$s = \sum_{i=1}^{n} b_i \chi_{A_i},$$

with $b_i \in B$ and disjoint $A_i \in \mathcal{A}$. A measurable function $f : \Omega \to B$ is ***Bochner integrable*** if there exists a sequence of integrable simple functions s_n such that

$$\lim_{n\to\infty}\int_\Omega \|f - s_n\|_B\,\mathrm{d}\mu = 0.$$

Then the ***Bochner integral*** is defined as

$$\int_\Omega f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega s_n \,\mathrm{d}\mu.$$

Lemma 1.7. *A measurable function f is Bochner integrable if and only if*

$$\int_\Omega \|f\| \,\mathrm{d}\mu < \infty.$$

Lemma 1.8. *Let B be a separable Hilbert space and D a countable, dense set of linear functionals of norm 1. Then it holds for all $x \in B$*

$$\|x\| = \sup_{f\in D} |f(x)|.$$

Definition 1.12. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and (ε_i) a sequence of independent Bernoulli random variables with probability $p = 1/2$. The ε_i can take the two values ± 1. Then the sequence (ε_i) is called a ***Rademacher sequence***.

Lemma 1.9. *Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be convex and D a countable, dense set of linear functionals of norm 1. Let further (X_i) be a finite sequence of independent random variables in B with $\mathbb{E}F(\|X_i\|) < \infty$ for all i. Let (ε_i) be a Rademacher sequence which is independent of (X_i). Then it holds*

$$\mathbb{E}F\left(\sup_{f\in D}\left|\sum_i f(X_i) - \mathbb{E}f(X_i)\right|\right) \le \mathbb{E}F\left(2\left\|\sum_i \varepsilon_i X_i\right\|\right)$$

and

$$\mathbb{E}F\left(\sup_{f\in D}\left|\sum_i \varepsilon_i\left(f(X_i) - \mathbb{E}f(X_i)\right)\right|\right) \le \mathbb{E}F\left(2\left\|\sum_i X_i\right\|\right).$$

1.3 Discrepancy

It is a natural question to ask for the uniformity of the distribution of a given point set in $[0,1]^d$ - and of course for the discrepancy between a perfectly uniformly distributed point set and this given point set. This question directly leads to the definition of the discrepancy function.

Definition 1.13. Let $B = B(x) = [0,x] \subset [0,1]^d$ and $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ be a point set. We define the ***discrepancy function*** of B and $\mathcal{P}$ by

$$D(B,\mathcal{P}) = \frac{1}{N}\sum_{j=1}^{N} \chi_B(t_j) - \mathrm{vol}(B). \tag{1.3.1}$$

The discrepancy function is a normalized measure for the deviation of the relative number of points of $\mathcal{P}$ in the box B from the optimal number $N\mathrm{vol}(B)$, which would be achieved by a perfectly uniformly distributed point set $\mathcal{P}$.

Sometimes, instead of $D(B,\mathcal{P})$, the discrepancy function is defined as $-D(B,\mathcal{P})$ or as $ND(B,\mathcal{P})$. It is obvious that

$$\frac{1}{N}\sum_{j=1}^{N}\chi_B(t_j) = \frac{1}{N}\cdot|\mathcal{P}\cap B|\,.$$

The following figure shows an example for the two-dimensional case.

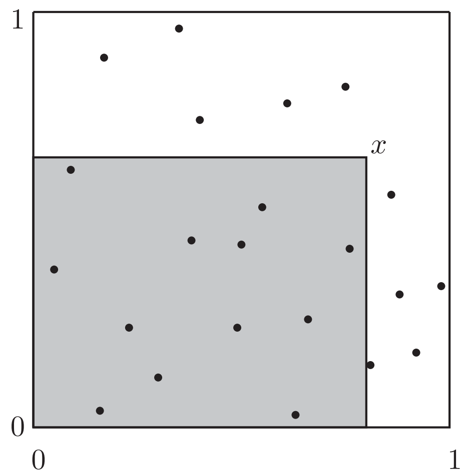

Figure 1.1: Point set of $N = 22$ points and box $B = [0, x]$ for $d = 2$

In this example for $d = 2$ we have $N = 22$ points. The box $B = [0, x]$ is given by $x = (0.8, 0.65)$. The relative number of points in B is $|\mathcal{P}\cap B| = 12$. This gives for the discrepancy function of B and $\mathcal{P}$ the value $D(B,\mathcal{P}) = 1/22\cdot 12 - 0.52 = 0.0255$.

It is possible to weight the points t_i by weights a_i, which leads to the definition of a weighted discrepancy.

Definition 1.14. Let $B = [0, x]$ and $\mathcal{P} = \{t_1,\dots,t_N\}\subset[0,1]^d$ be a point set. Let further $a = (a_1,\dots,a_N)\in\mathbb{R}^N$. We define the ***a-weighted discrepancy function*** of B and $\mathcal{P}$ by

$$D(a,B,\mathcal{P}) = \sum_{j=1}^{N} a_j\chi_B(t_j) - \mathrm{vol}(B). \tag{1.3.2}$$

Usually, one is interested in the norm of the discrepancy function in special function spaces. In this context, the case of the L_p-spaces is most frequently studied.

Definition 1.15. Let $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ be a point set. For $0 < p < \infty$ we define the ***L_p-star discrepancy*** of $\mathcal{P}$ by

$$D_p^*(\mathcal{P}) = \left(\int_{[0,1]^d} |D(B,\mathcal{P})|^p \, \mathrm{d}x \right)^{1/p}. \tag{1.3.3}$$

For $p = \infty$ we define the ***star discrepancy*** of $\mathcal{P}$ by

$$D_\infty^*(\mathcal{P}) = \sup_{x \in [0,1]^d} |D(B,\mathcal{P})|. \tag{1.3.4}$$

Definition 1.16. Let $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$. For $0 < p < \infty$, we define the ***a-weighted L_p-star discrepancy*** of $\mathcal{P}$ by

$$D_p^*(a,\mathcal{P}) = \left(\int_{[0,1]^d} |D(a,B,\mathcal{P})|^p \, \mathrm{d}x \right)^{1/p}. \tag{1.3.5}$$

For $p = \infty$ we define the ***a-weighted star discrepancy*** of $\mathcal{P}$ by

$$D_\infty^*(a,\mathcal{P}) = \sup_{x \in [0,1]^d} |D(a,B,\mathcal{P})|. \tag{1.3.6}$$

Obviously, for $a_j = \frac{1}{N}$ we obtain the standard L_p-star discrepancy. For convenience, we call the a-weighted L_p-star discrepancy simply a-weighted L_p-discrepancy or weighted L_p-discrepancy.

The following result was presented by Warnock in [52].

Lemma 1.10. *For the a-weighted L_2-discrepancy of a point set $\mathcal{P} = \{t_1, \ldots, t_N\}$ holds*

$$D_2^*(a,P)^2 = \frac{1}{3^d} - \frac{1}{2^{d-1}} \sum_{j=1}^N a_j \prod_{k=1}^d \left(1 - t_{j,k}^2\right) + \sum_{i,j=1}^N a_i a_j \prod_{k=1}^d \left(1 - \max\{t_{i,k}, t_{j,k}\}\right). \tag{1.3.7}$$

One of the major goals in discrepancy theory is to find point sets with best possible discrepancy or to prove that there exists a point set with low discrepancy. For that purpose we define minimal discrepancies.

Definition 1.17. For $0 < p < \infty$, the ***minimal L_p-star discrepancy*** is defined by

$$D_p^*(N,d) = \inf_{\substack{\mathcal{P} \subset [0,1]^d \\ \#\mathcal{P} = N}} D_p^*(\mathcal{P}). \tag{1.3.8}$$

The ***minimal star discrepancy*** is defined by

$$D^*(N,d) = \inf_{\substack{\mathcal{P}\subset[0,1]^d \\ \#\mathcal{P}=N}} D^*_\infty(\mathcal{P}). \tag{1.3.9}$$

Let $\varepsilon > 0$. The ***inverse of the star discrepancy*** is defined as

$$N^*(d,\varepsilon) = \min\left\{N : D^*(N,d) \leq \varepsilon\right\}. \tag{1.3.10}$$

Since we want to study different types of discrepancies it is useful to define a general discrepancy. Therefore, let $(\Omega_d, \mathcal{A}_d, \mu_d)$ be a probability space. For each fixed $x \in \Omega_d$ we consider one measurable subset $B(x) \subset \Omega_d$. Furthermore, we claim that the mapping $(t,x) \mapsto \chi_{B(x)}(t)$ is also measurable.

Definition 1.18. For a point set $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ the ***B-discrepancy*** is defined as

$$D^B(B(x),\mathcal{P}) = \frac{1}{N}\sum_{i=1}^N \chi_{B(x)}(t_i) - \lambda^d(B(x)). \tag{1.3.11}$$

The ***L_p-B-discrepancy*** is defined as

$$D^B_p(\mathcal{P}) = \left(\int_{\Omega_d} \left|D^B(B(x),\mathcal{P})\right|^p \,\mathrm{d}\mu_d(x)\right)^{1/p}. \tag{1.3.12}$$

This definition is similar to the L_p-B-discrepancy defined by Novak and Woźniakowski in [41]. While they use densities, we use measures. If the measure μ_d is absolutely continuous with respect to the Lebesgue measure, we obtain the definition of Novak and Woźniakowski via the Radon-Nikodym Theorem.

The L_2-B-discrepancy was generalized to a weighted geometric L_2-discrepancy by Gnewuch in [26].

The major goal of this work is to study expected discrepancies. For that purpose, we finally need the following definition.

Definition 1.19. Let $0 < p < \infty$. The ***average L_p-star discrepancy*** is defined as

$$\mathrm{av}^*_p(N,d) = \left(\mathbb{E}\left(D^*_p(\mathcal{P})^p\right)\right)^{1/p}. \tag{1.3.13}$$

The expectation is taken over sets $\mathcal{P}$ of N independent and uniformly distributed points $t_1, \ldots, t_N \in [0,1]^d$. Further, we define the ***average L_p-B-discrepancy*** as

$$\mathrm{av}^B_p(N,d) = \left(\mathbb{E}\left(D^B_p(\mathcal{P})^p\right)\right)^{1/p}. \tag{1.3.14}$$

For detailed information about discrepancy we refer to [4], [16], [21], [30], [39] and [41].

2 Limit behavior of average L_p-discrepancies

2.1 Known results about average L_p-discrepancies

This chapter is closely related to a paper of Heinrich, Novak, Wasilkowski and Woźniakowski from 2001 ([30]). In this essential paper for discrepancy theory, the authors investigate the inverse of the star-discrepancy $N^*(d, \varepsilon)$. They show that for the inverse of the star discrepancy the upper bound

$$N^*(\varepsilon, d) \leq C d \varepsilon^{-2} \tag{2.1.1}$$

holds, where $C > 0$. To prove this upper bound they use probabilistic methods. Because of the unknown constant C, this term can not be computed for explicit values of ε and d. Thus, the authors introduced two other bounds for $N^*(n, d)$ with known constants, namely

$$N^*(\varepsilon, d) \leq C_k d^2 \varepsilon^{-2-1/k} \quad \text{for } k = 1, 2, \ldots$$

and

$$N^*(\varepsilon, d) = O(d\varepsilon^{-2}(\log d + \log \varepsilon^{-1})).$$

To prove the first one, the authors use a technique which is based on the analysis of the average L_p-star discrepancy $\mathrm{av}_p^*(N, d)$. For even p they compute an explicit expression for the average L_p-star discrepancy

$$\mathrm{av}_p^*(N, d)^p = \sum_{r=p/2}^{p-1} C(r, p, d) N^{-r}, \tag{2.1.2}$$

with known constants $C(r, p, d)$, which depend on Stirling numbers of the first and second kind. Because the explicit expression for $\mathrm{av}_p(N, d)$ is a sum of alternating terms, it is hard to handle. Thus, the authors show the upper bound

$$\mathrm{av}_p^*(N, d) \leq 3^{2/3} 2^{5/2+d/p} p(p+2)^{-d/p} N^{-1/2},$$

with p again even. To improve this bound, Hinrichs and Novak used symmetrization ([31]). This technique yields an expression with only positive summands for the average L_p-star discrepancy and leads to

$$\begin{aligned} \mathrm{av}_p^*(N,d) &\le 2^{1/2+d/p} p^{1/2} (p+2)^{-d/p} N^{-1/2}, && \text{for } p \ge 2d, \\ \mathrm{av}_p^*(N,d) &\le 2^{3/2-d/p} N^{-1/2}, && \text{for } p < 2d. \end{aligned}$$

This idea of symmetrization was further applied by Gnewuch ([25]). He computed bounds for the average L_p-extreme discrepancy $\mathrm{av}_p(N,d)$. To get this type of discrepancy axis-parallel boxes in $[-1,1]^d$ instead of boxes in $[0,1]^d$ anchored in the origin are studied. Gnewuch used symmetrization and rather simple combinatorial arguments to get the bounds

$$\begin{aligned} \mathrm{av}_p(N,d) &\le 2^{1/2+3d/p} p^{1/2} (p+2)^{-d/p} (p+4)^{-d/p} N^{-1/2}, && \text{for } p \ge 4d, \\ \mathrm{av}_p(N,d) &\le 2^{5/4} 3^{1/4-d} N^{-1/2}, && \text{for } p < 4d. \end{aligned}$$

Bounds for general $p \in [2,\infty)$ can be obtained by using Hölder's inequality (see e.g. Gnewuch [25]).

Recently, Aistleitner proved (2.1.1) with the constant $C = 100$ ([1]). Furthermore, there exists also a lower bound for the inverse of the star-discrepancy

$$N^*(\varepsilon,d) \ge \widetilde{C}\,\frac{d}{\varepsilon}, \text{ with } 0 < \varepsilon < \varepsilon_0$$

which was proved by Hinrichs in [32].

This chapter is motivated by a work of Steinerberger ([50]), who showed for arbitrary $p > 0$ the limit relation

$$\lim_{N\to\infty} \mathrm{av}_p^*(N,d)^p N^{p/2} = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int_{[0,1]^d} \left[\lambda^d([\mathbf{0},\mathbf{x}))\Big(1-\lambda^d([\mathbf{0},\mathbf{x}))\Big)\right]^{p/2} \mathrm{d}\mathbf{x}. \tag{2.1.3}$$

In order to interpret the meaning of the above equation, three important aspects need to be mentioned. First of all, we have an expression for arbitrary p. Previous results mostly gave expressions only for even p. Another aspect is that we can use this result to argue, why the sum (2.1.2) has to start from $p/2$ instead of one. Heinrich, Novak, Wasilkowski and Woźniakowski present a rather complicated proof to show this in [30]. Finally, (2.1.3) yields another expression for the first summand of (2.1.2). Consequently, we get a relation between Stirling numbers and the right hand side of the result of Steinerberger. Though, it needs to be stated that in order to apply this result, explicit expressions for fixed N and d are needed. Moreover, this result gives no quantitative estimates for the speed of convergence. Getting such bounds is more complicated.

2.2 Limit behavior of average L_p-discrepancies for arbitrary p

In this section we present results of Hinrichs and Weyhausen ([35]).

Let $(\Omega_d, 2^{\Omega_d}, \mu_d)$ be a probability space. For convenience we simply write (Ω_d, μ_d).

First we show a lemma which yields an elementary upper bound for the L_p-B-discrepancy. The idea is symmetrization of random variables and was first used in this context by Hinrichs and Novak in [31] for the L_p-star discrepancy.

Lemma 2.1. *Let (Ω_d, μ_d) be a probability space and p even. We have*

$$\mathrm{av}_p^B(N,d)^p \le c(p) N^{-p/2}. \tag{2.2.1}$$

Proof. We define random variables $X_i : [0,1]^{Nd} \to L_p(\mu_d)$ by

$$X_i(t)(.) = \chi_{B(.)}(t_i)$$

with $t = (t_1, \dots, t_N) \in [0,1]^{Nd}$. These random variables are Bochner integrable and

$$\mathbb{E}_t X_i(t)(x) = \lambda^d(B(x)),$$

where $\mathbb{E}_t$ is the expected value with respect to uniformly distributed $t \in [0,1]^{Nd}$ for fixed $x \in \Omega_d$.

We get for the average L_p-B-discrepancy

$$\begin{aligned}
\mathrm{av}_p^B(N,d)^p &= \int\limits_{[0,1]^{Nd}} \int\limits_{\Omega_d} \left| \lambda^d(B(x)) - \frac{1}{N} \sum_{i=1}^N \chi_{B(x)}(t_i) \right|^p \mathrm{d}\mu_d(x)\,\mathrm{d}t \\
&= \int\limits_{[0,1]^{Nd}} \int\limits_{\Omega_d} \left| \frac{1}{N} \sum_{i=1}^N \Big(\mathbb{E}_t X_i(t)(x) - X_i(t)(x) \Big) \right|^p \mathrm{d}\mu_d(x)\,\mathrm{d}t \\
&= \int\limits_{[0,1]^{Nd}} \left\| \frac{1}{N} \sum_{i=1}^N \Big(\mathbb{E}_t X_i(t) - X_i(t) \Big) \Big| L_p(\mu_d) \right\|^p \mathrm{d}t \\
&= \mathbb{E}_t \left\| \frac{1}{N} \sum_{i=1}^N \Big(X_i(t) - \mathbb{E}_t X_i(t) \Big) \Big| L_p(\mu_d) \right\|^p .
\end{aligned} \tag{2.2.2}$$

In order to compute this norm we apply Lemma 1.8. Let D be a countable dense subset of the unit ball of the dual space of $L_p(\mu_d)$. Hence, we have for all $X \in L_p(\mu_d)$ the equality $\|X \mid L_p(\mu_d)\| = \sup_{f\in D} |f(X)|$. This leads to the following expression

$$\begin{aligned}
\left\| \sum_{i=1}^{N} \Big(X_i(t) - \mathbb{E}_t X_i(t) \Big) \Big| L_p(\mu_d) \right\| &= \sup_{f\in D} \left| f\left(\sum_{i=1}^{N} \Big(X_i(t) - \mathbb{E}_t X_i(t) \Big) \right) \right| \\
&= \sup_{f\in D} \left| \sum_{i=1}^{N} \Big(f\Big(X_i(t)\Big) - f\Big(\mathbb{E}_t X_i(t)\Big) \Big) \right| \\
&= \sup_{f\in D} \left| \sum_{i=1}^{N} \Big(f\Big(X_i(t)\Big) - \mathbb{E}_t f\Big(X_i(t)\Big) \Big) \right|. \qquad (2.2.3)
\end{aligned}$$

Because the random variables X_i are Bochner integrable, we can apply Lemma 1.9 and replace the supremum over all functionals $f \in D$ by an expression which depends on symmetric Rademacher random variables $\varepsilon_1, \ldots, \varepsilon_N : \Omega' \to \{-1, +1\}$, independent of $\{X_i\}$. Namely, we get

$$\begin{aligned}
\mathrm{av}_p^B(N,d)^p &\overset{(2.2.2)}{=} \mathbb{E}_t \left\| \frac{1}{N} \sum_{i=1}^{N} \Big(X_i(t) - \mathbb{E}_t X_i(t) \Big) \Big| L_p(\mu_d) \right\|^p \\
&\overset{(2.2.3)}{=} \mathbb{E}_t \left(\sup_{f\in D} \left| \frac{1}{N} \sum_{i=1}^{N} \Big(f\Big(X_i(t)\Big) - \mathbb{E}_t f\Big(X_i(t)\Big) \Big) \right| \right)^p \\
&\leq \mathbb{E}_{t,\varepsilon} \left(2 \left\| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i X_i \Big| L_p(\mu_d) \right\| \right)^p \\
&= \left(\frac{2}{N}\right)^p \mathbb{E}_{t,\varepsilon} \left(\int_{\Omega_d} \left(\sum_{i=1}^{N} \varepsilon_i X_i(t)(x) \right)^p \mathrm{d}\mu_d(x) \right) \\
&= \left(\frac{2}{N}\right)^p \mathbb{E}_{t,\varepsilon} \left(\int_{\Omega_d} \sum_{i_1,\ldots,i_p=1}^{N} \prod_{l=1}^{p} \varepsilon_{i_l} X_{i_l}(t)(x) \, \mathrm{d}\mu_d(x) \right) \\
&= \left(\frac{2}{N}\right)^p \sum_{i_1,\ldots,i_p=1}^{N} \mathbb{E}_{t,\varepsilon} \left(\int_{\Omega_d} \prod_{l=1}^{p} \varepsilon_{i_l} \prod_{l=1}^{p} X_{i_l}(t)(x) \, \mathrm{d}\mu_d(x) \right) \\
&= \left(\frac{2}{N}\right)^p \sum_{i_1,\ldots,i_p=1}^{N} \mathbb{E}_{t,\varepsilon} \left(\prod_{l=1}^{p} \varepsilon_{i_l} \int_{\Omega_d} \prod_{l=1}^{p} X_{i_l}(t)(x) \, \mathrm{d}\mu_d(x) \right) \\
&= \left(\frac{2}{N}\right)^p \sum_{i_1,\ldots,i_p=1}^{N} \mathbb{E}_{\varepsilon} \left(\prod_{l=1}^{p} \varepsilon_{i_l} \right) \mathbb{E}_t \left(\int_{\Omega_d} \prod_{l=1}^{p} X_{i_l}(t)(x) \, \mathrm{d}\mu_d(x) \right). \qquad (2.2.4)
\end{aligned}$$

In order to estimate (2.2.4), we consider for $k \in [p] = \{1, 2, \dots, p\}$ pairwise disjoint indices $i_1, \dots, i_k \in [N]$ and multiplicities $j_1, \dots, j_k \in [p]$ with $\sum_{l=1}^{k} j_l = p$ (see Gnewuch, [25]). We define

$$\begin{aligned} J &= \mathbb{E}_\varepsilon \left(\prod_{l=1}^{k} \varepsilon_{i_l}^{j_l} \right) \mathbb{E}_t \left(\int_{\Omega_d} \prod_{l=1}^{k} X_{i_l}^{j_l}(t)(x) \, \mathrm{d}\mu_d(x) \right) \\ &= \prod_{l=1}^{k} \mathbb{E}_\varepsilon \left(\varepsilon_{i_l}^{j_l} \right) \mathbb{E}_t \left(\int_{\Omega_d} \prod_{l=1}^{k} X_{i_l}^{j_l}(t)(x) \, \mathrm{d}\mu_d(x) \right). \end{aligned}$$

If there exists at least one odd exponent j_l, the first factor is zero. If all exponents j_l are even, then the first factor is one. Especially $J = 0$ if $k > p/2$.

Furthermore, let $T(p, k, N)$ be the number of tuples $(i_1, \dots, i_p) \in [N]^p$ which fulfill $|\{i_1, \dots, i_p\}| = k$ and $|\{l \in [p] : i_l = i_m\}|$ even for each $m \in [p]$.

With this notation we get

$$\begin{aligned} \mathrm{av}_p^B(N, d)^p &\le \left(\frac{2}{N} \right)^p \sum_{k=1}^{p/2} T(p, k, N) \mathbb{E}_t \left(\int_{\Omega_d} \prod_{l=1}^{k} X_l(t)(x) \, \mathrm{d}\mu_d(x) \right) \\ &\le \left(\frac{2}{N} \right)^p \sum_{k=1}^{p/2} T(p, k, N). \end{aligned} \tag{2.2.5}$$

Using the numbers $\#(p, k, N)$, which are well known in combinatorics, we can estimate the numbers $T(p, k, N)$ by

$$T(p, k, N) \le \#(p, k, N).$$

This $\#(p, k, N)$ is defined as the cardinality of the set of tuples $(i_1, \dots, i_p) \in [N]^p$ with $|\{i_1, \dots, i_p\}| = k$ and can be expressed by Stirling numbers of the second kind (see Heinrich, Novak, Wasilkowski and Woźniakowski, [30]) by

$$\#(p, k, N) = \binom{N}{k} k! \, S(p, k).$$

Hence, we get

$$T(p, k, N) \le \binom{N}{k} k! \, S(p, k) \le \binom{N}{k} (p/2)! \, S(p, p/2) \le c(k, p) N^k \le c(k, p) N^{p/2}$$

which, together with (2.2.5), yields the result. $\square$

Now we present a result of Steinerberger ([50]) and fill a gap in his proof.

Theorem 2.1 (Steinerberger)**.** *Let $p > 0, d \in \mathbb{N}$. Then*

$$\lim_{N\to\infty} N^{p/2} \mathrm{av}_p^*(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int\limits_{[0,1]^d} \left[\lambda^d([0,x))\left(1-\lambda^d([0,x))\right)\right]^{p/2} \mathrm{d}x. \tag{2.2.6}$$

We give no proof of this result yet. Instead we present a result, which contains Theorem 2.1 as a special case.

Theorem 2.2. *Let $p > 0, d \in \mathbb{N}$, further let (Ω_d, μ_d) be a probability space and $\{B(x) : x \in \Omega_d\} \subset 2^{[0,1]^d}$ the allowed sets. Then*

$$\lim_{N\to\infty} N^{p/2} \mathrm{av}_p^B(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int\limits_{\Omega_d} \left[\lambda^d(B(x))\left(1-\lambda^d(B(x))\right)\right]^{p/2} \mathrm{d}\mu_d(x). \tag{2.2.7}$$

Proof. Switching the order of integration we get

$$\begin{aligned}
\mathrm{av}_p^B(N,d)^p &= \int\limits_{[0,1]^{Nd}} \int\limits_{\Omega_d} \left| \lambda^d(B(x)) - \frac{1}{N}\sum_{i=1}^N \chi_{B(x)}(t_i) \right|^p \mathrm{d}\mu_d(x)\,\mathrm{d}t \\
&= \int\limits_{\Omega_d} \int\limits_{[0,1]^{Nd}} \left| \lambda^d(B(x)) - \frac{1}{N}\sum_{i=1}^N \chi_{B(x)}(t_i) \right|^p \mathrm{d}t\,\mathrm{d}\mu_d(x).
\end{aligned}$$

Now, we want to have a closer look at the inner integral.

Therefore, we interpret for fixed $x \in \Omega_d$ the characteristic functions $\chi_{B(x)}(t_i)$ as Bernoulli random variables $X_i : [0,1]^{Nd} \to \{0,1\}$ with probability $\lambda = \lambda^d(B(x))$, where we first assume $\lambda \neq 0, 1$. Their expected value is $\mathbb{E}(X_i) = \lambda$ and their variance is $\sigma^2(X_i) = \lambda(1-\lambda)$. Hence, the sum $\sum_{i=1}^N X_i$ is binomial distributed with expected value $\mathbb{E}\left(\sum_{i=1}^N X_i\right) = N\lambda$ and variance $\sigma^2\left(\sum_{i=1}^N X_i\right) = N\lambda(1-\lambda)$. The central limit Theorem (Theorem 1.10) now gives for fixed $x \in \Omega_d$ with $\lambda^d(B(x)) \neq 0, 1$

$$X_{N,\lambda^d(B(x))} = \left(\lambda - \frac{1}{N}\sum_{i=1}^N X_i\right)\sqrt{N} \xrightarrow{\mathcal{D}} f(\lambda)Y, \tag{2.2.8}$$

with $Y \sim \mathcal{N}(0,1)$ and $f(\lambda) = \sqrt{\lambda(1-\lambda)}$. Observe, that (2.2.8) holds obviously for $\lambda = 1$ and $\lambda = 0$ too.

This is only a pointwise convergence for fixed x. Because there is no uniform convergence given, it is not enough to integrate over $x \in \Omega_d$ to get the result.

Instead, we will use the following approach. Let Λ be a random variable on the probability space (Ω_d, μ_d), given by

$$\Lambda(x) = \lambda^d(B(x))$$

and independent of Y. Now, $X_{N,\Lambda}$ is a random variable obtained by first choosing λ according to the distribution of Λ and then using $X_{N,\lambda^d(B(x))}$. Then

$$N^{p/2}\mathrm{av}_p^B(N,d)^p = \mathbb{E}\left|X_{N,\Lambda}\right|^p.$$

We will show the equation

$$\lim_{N\to\infty} \mathbb{E}\left|X_{N,\Lambda}\right|^p = \mathbb{E}\left|f(\Lambda)Y\right|^p = \mathbb{E}f(\Lambda)^p\,\mathbb{E}\left|Y\right|^p. \tag{2.2.9}$$

This finally yields the result

$$\begin{aligned}\lim_{N\to\infty} N^{p/2}\mathrm{av}_p^B(N,d)^p &= \mathbb{E}|Y|^p \int_{\Omega_d} \left[\lambda^d(B(x))\Big(1-\lambda^d(B(x))\Big)\right]^{p/2} \mathrm{d}\mu_d(x) \\ &= \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int_{\Omega_d}\left[\lambda^d(B(x))\Big(1-\lambda^d(B(x))\Big)\right]^{p/2} \mathrm{d}\mu_d(x).\end{aligned}$$

We will use the characteristic functions of $X_{N,\Lambda}$ and $f(\Lambda)Y$, given by

$$\begin{aligned}\varphi_{X_{N,\Lambda}}(s) = \mathbb{E}e^{isX_{N,\Lambda}} &= \int_\Omega e^{isX_{N,\Lambda}}\,\mathrm{d}\mathbb{P} = \int_{\Omega_d}\mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}}\,\mathrm{d}\mu_d(x) \quad \text{and}\\ \varphi_{f(\Lambda)Y}(s) = \mathbb{E}e^{isf(\Lambda)Y} &= \int_\Omega e^{isf(\Lambda)Y}\,\mathrm{d}\mathbb{P} = \int_{\Omega_d}\mathbb{E}_t e^{isf(\lambda^d(B(x)))Y}\,\mathrm{d}\mu_d(x),\end{aligned}$$

for $s \in \mathbb{R}$. We start with equation (2.2.8) and use the definition of convergence in distribution. Because the exponential function is bounded and continuous, we get for fixed $s \in \mathbb{R}$ and $x \in [0,1]^d$ and therefore fixed $\lambda = \lambda^d(B(x))$ the equation

$$\lim_{N\to\infty}\mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}} = \mathbb{E}_t e^{isf(\lambda)Y}. \tag{2.2.10}$$

Furthermore, we have

$$\left|\mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}}\right| \le g(x) = 1 \in L_1(\mu_d).$$

Thus, we can apply the dominated convergence Theorem (Theorem 1.9) and change the order of limit and integral. This gives

$$\lim_{N\to\infty} \int_{\Omega_d} \mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}} \,\mathrm{d}\mu_d = \int_{\Omega_d} \lim_{N\to\infty} \mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}} \,\mathrm{d}\mu_d. \tag{2.2.11}$$

Combining the definition of the characteristic functions of $X_{N,\Lambda}$ and $f(\Lambda)Y$ with the equations (2.2.10) and (2.2.11), we obtain

$$\begin{aligned}
\lim_{N\to\infty} \varphi_{X_{N,\Lambda}}(s) &= \lim_{N\to\infty} \int_{\Omega_d} \mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}} \,\mathrm{d}\mu_d(x) \\
&= \int_{\Omega_d} \lim_{N\to\infty} \mathbb{E}_t e^{isX_{N,\lambda^d(B(x))}} \,\mathrm{d}\mu_d(x) \\
&= \int_{\Omega_d} \mathbb{E}_t e^{isf(\lambda^d(B(x)))Y} \,\mathrm{d}\mu_d(x) \\
&= \varphi_{f(\Lambda)Y}(s) \qquad (2.2.12)
\end{aligned}$$

for $s \in \mathbb{R}$. Now, the Lévy-Cramér continuity Theorem (Theorem 1.6) gives

$$X_{N,\Lambda} \xrightarrow{\mathcal{D}} f(\Lambda)Y. \tag{2.2.13}$$

Lemma 2.1 gives that $\mathrm{av}_p^B(N,d)^p$ is of order $N^{-p/2}$ for even p. Thus, we have for every even p

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left(|X_{N,\Lambda}|^p\right) = \sup_{N\in\mathbb{N}} N^{p/2}\,\mathrm{av}_p^B(N,d)^p \le \sup_{N\in\mathbb{N}} N^{p/2}\,N^{-p/2}c(p) < \infty.$$

Hence, we can apply Lemma 1.6 and finally get

$$\lim_{N\to\infty} \mathbb{E}\,|X_{N,\Lambda}|^p = \mathbb{E}\,|f(\Lambda)Y|^p$$

for every $0 < p < \infty$, which completes the proof. □

If we choose $\Omega_d = [0,1]^d$, $\mu_d = \lambda^d$ and $B(x) = [0,x)$ we obtain the average L_p-star discrepancy and with it Theorem 2.1.

Remark 2.1. As already mentioned, the proof of Theorem 2.1 given by Steinerberger in [50] is not complete. He claimed, that the Berry-Esseen Theorem would suffice to prove Theorem 2.1 We will show here that the Berry-Esseen Theorem is not strong enough.

For that purpose, we use the same notation as in Theorem 2.2 but just consider the L_p-star discrepancy. Hence, we have for fixed $x \in [0,1]^d$ the Bernoulli random variables

$$X_i = \chi_{[0,x)}(t_i)$$

with expected value $\mu = \lambda = \lambda^d([0,x))$, variance $\sigma^2 = \lambda(1-\lambda)$ and third absolute central moment $\rho = \lambda(1-\lambda)(\lambda^2 + (1-\lambda)^2)$. Now the Berry-Esseen Theorem (Theorem 1.11) gives us for the standardized random variable

$$Z_N = \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} = \frac{X_1 + \cdots + X_N - N\lambda}{\sqrt{\lambda(1-\lambda)}\sqrt{N}}$$

the inequality

$$|F_{Z_N}(x) - \Phi_{0,1}(x)| \leq \frac{c\,(\lambda^2 + (1-\lambda)^2)}{\sqrt{\lambda(1-\lambda)}\sqrt{N}} = c_1(N,\lambda),$$

with a constant $c \leq 0.8$ and for all $x \in \mathbb{R}$. With F_{Z_N} we constitute the distribution function of Z_N and with $\Phi_{0,1}$ the distribution function of the standard normal distribution. This yields

$$\left|\mathbb{P}\left(|X| > y\right) - \mathbb{P}\left(|Y| > \sqrt{\frac{N}{\lambda(1-\lambda)}}\,y\right)\right| \leq 2c_1(N,\lambda) + \mathbb{P}(X = -y) \tag{2.2.14}$$

with

$$X = \frac{X_1 + \cdots + X_N - N\lambda}{N}$$

and Y standard normal distributed. We just look at the upper bound and get

$$\begin{aligned}
\mathbb{E}\,|X|^p &= p\int_0^1 \mathbb{P}\left(|X| > y\right) y^{p-1}\,\mathrm{d}y \\
&\overset{(2.2.14)}{\leq} p\left(\int_0^1 \left(\mathbb{P}\left(|Y| > \sqrt{\frac{N}{\lambda(1-\lambda)}}\,y\right) + 2c_1(N,\lambda) + \mathbb{P}(X = -y)\right) y^{p-1}\,\mathrm{d}y\right) \\
&\leq p\int_0^\infty \mathbb{P}(|Y| > x)\left(\frac{\lambda(1-\lambda)}{N}\right)^{p/2} x^{p-1}\,\mathrm{d}x + 2c_1(N,\lambda) \\
&= \left(\frac{\lambda(1-\lambda)}{N}\right)^{p/2} \mathbb{E}\,|Y|^p + 2c_1(N,\lambda) \\
&= \left(\frac{\lambda(1-\lambda)}{N}\right)^{p/2} \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{p+1}{2}\right) + 2\,\frac{c\,(\lambda^2 + (1-\lambda)^2)}{\sqrt{\lambda(1-\lambda)}\sqrt{N}}.
\end{aligned}$$

Multiplying now first with $N^{p/2}$ then integrate over x and taking the limit in N we get

$$\begin{aligned}
&\lim_{N\to\infty} N^{p/2}\mathrm{av}_p^*(N,d)^p \\
&\qquad = \lim_{N\to\infty} N^{p/2} \int\limits_{[0,1]^d} \mathbb{E}\,|X|^p \,\mathrm{d}x \\
&\qquad \le \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{p+1}{2}\right) \int\limits_{[0,1]^d} \Big[\lambda^d([0,x))\big(1-\lambda^d([0,x))\big)\Big]^{p/2} \mathrm{d}x \\
&\qquad\quad + 2c \lim_{N\to\infty} N^{p/2-1/2} \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\le\lambda^d([0,x))\le 1-\varepsilon} \frac{\left(\lambda^d([0,x))^2 + \big(1-\lambda^d([0,x))\big)^2\right)}{\sqrt{\lambda^d([0,x))\big(1-\lambda^d([0,x))\big)}} \mathrm{d}x.
\end{aligned}$$

Obviously, the second summand is infinite for all $p > 1$. Hence, we get no asymptotic expression for the L_p-star discrepancy using the Berry-Esseen Theorem.

In the following lemma we adopt some results of Steinerberger [50] to show how the integral in Theorem 2.2 can be computed and estimated.

Lemma 2.2 (Steinerberger)**.** *Let $p > 0, d \in \mathbb{N}$, let further (Ω_d, μ_d) be a probability space and $\{B(x) : x \in \Omega_d\} \subset 2^{[0,1]^d}$ the allowed sets. Let*

$$I = \int\limits_{\Omega_d} \Big[\lambda^d(B(x))\big(1-\lambda^d(B(x))\big)\Big]^{p/2} \mathrm{d}\mu_d(x)$$

Then

(i) $I = \sum_{k=0}^{p/2}(-1)^k\binom{p/2}{k} \int\limits_{\Omega_d} \big(\lambda^d(B(x))\big)^{p/2+k} \mathrm{d}\mu_d(x)$ *for even p,*

(ii) $I = \sum_{k=0}^{\infty}(-1)^k\binom{p/2}{k} \int\limits_{\Omega_d} \big(\lambda^d(B(x))\big)^{p/2+k} \mathrm{d}\mu_d(x),$

(iii) $I \le \int\limits_{\Omega_d} \big(\lambda^d(B(x))\big)^{p/2} \mathrm{d}\mu_d(x),$

(iv) $I \ge \int\limits_{\Omega_d} \big(\lambda^d(B(x))\big)^{p/2} - (2^{p/2}-1)\big(\lambda^d(B(x))\big)^{p/2+1} \mathrm{d}\mu_d(x),$

(v) $I \ge \int\limits_{\Omega_d} \big(\lambda^d(B(x))\big)^{p/2} - \frac{p}{2}\big(\lambda^d(B(x))\big)^{p/2+1} \mathrm{d}\mu_d(x)$ *for $p \ge 2$.*

Proof. (i) We apply the binomial Theorem.

(ii) We apply the generalized binomial Theorem (see e.g. [28] , page 162)

(iii) We use the trivial estimate $\lambda^d(B(x)) \geq 0$.

(iv) We estimate

$$\begin{aligned}\Big[\lambda^d(B(x))\big(1-\lambda^d(B(x))\big)\Big]^{p/2} &= \sum_{k=0}^{\infty}(-1)^k\binom{p/2}{k}\Big(\lambda^d(B(x))\Big)^{p/2+k}\\ &\geq \Big(\lambda^d(B(x))\Big)^{p/2}-\sum_{k=1}^{\infty}\binom{p/2}{k}\Big(\lambda^d(B(x))\Big)^{p/2+k}\\ &\geq \Big(\lambda^d(B(x))\Big)^{p/2}-\left[\sum_{k=1}^{\infty}\binom{p/2}{k}\right]\Big(\lambda^d(B(x))\Big)^{p/2+1}\\ &= \Big(\lambda^d(B(x))\Big)^{p/2}-\Big(2^{p/2}-1\Big)\Big(\lambda^d(B(x))\Big)^{p/2+1}.\end{aligned}$$

(v) For $p \geq 2$ we define for fixed $x \in \mathbb{R}$ the function

$$f(t) = (x-t)^{p/2}$$

for $t \in [0, x]$ and use the mean value Theorem as well as the monotony of the first derivative of f. We get

$$(x-x^2)^{p/2} \geq x^{p/2} - \frac{p}{2}x^{p/2+1}.$$

Choosing $x = \lambda^d(B(x))$ yields the result.

□

Remark 2.2. The first expression is an alternating sum, which we know from other works (see Heinrich, Novak, Wasilkowski, Woźniakowski [30]). The second expression is an infinite series and therefore not easy to interpret. This is the reason why we stated the estimates, which were introduced by Steinerberger [50]. In order to compute these estimates, we have to solve integrals of the form

$$\int_{\Omega_d} \Big(\lambda^d(B(x))\Big)^k \,\mathrm{d}\mu_d(x) \tag{2.2.15}$$

for $k > 0$.

Now we use Theorem 2.2 for different types of discrepancies and compute integrals of the form (2.2.15).

Example 2.1 (L_p-discrepancy anchored in α)**.** To get $\mathrm{av}_p^{*,\alpha}(N,d)$, the average L_p-discrepancy anchored in $\alpha \in [0,1]^d$, we choose

$$\Omega_d = [0,1]^d \text{ and } \mu_d = \lambda^d.$$

The boxes $B(x)$ for fixed $x \in \Omega_d$ are defined as

$$B(x) = \bigtimes_{i=1}^{d} \Big[\min\{x_i, \alpha_i\}, \max\{x_i, \alpha_i\} \Big).$$

The following picture illustrates the boxes B for different x.

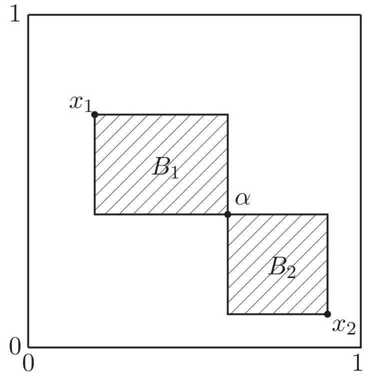

Figure 2.1: L_p-discrepancy anchored in $\alpha = [0.6, 0.4]$. Boxes B_1, B_2 for points $x_1, x_2 \in \Omega_2 = [0,1]^2$

These boxes have the Lebesgue measure

$$\lambda^d(B(x)) = \prod_{i=1}^{d} |x_i - \alpha_i| \,.$$

Theorem 2.2 gives

$$\lim_{N\to\infty} N^{p/2} \mathrm{av}_p^{*,\alpha}(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int\limits_{[0,1]^d} \left[\prod_{i=1}^{d} |x_i - \alpha_i| \left(1 - \prod_{i=1}^{d} |x_i - \alpha_i|\right)\right]^{p/2} \mathrm{d}x.$$

For $k > 0$ we compute the integral (2.2.15) and get

$$\int\limits_{[0,1]^d} \left(\prod_{i=1}^d |x_i - \alpha_i|\right)^k \mathrm{d}x = \prod_{i=1}^d \left(\int\limits_0^1 |x_i - \alpha_i|^k \, \mathrm{d}x_i\right)$$
$$= \prod_{i=1}^d \left(\int\limits_0^{\alpha_i} (\alpha_i - x_i)^k \, \mathrm{d}x_i + \int\limits_{\alpha_i}^1 (x_i - \alpha_i)^k \, \mathrm{d}x_i\right)$$
$$= \left(\frac{1}{k+1}\right)^d \prod_{i=1}^d \left(\alpha_i^{k+1} + (1-\alpha_i)^{k+1}\right).$$

If we choose $\alpha = 0$ we get the boxes $[0, x)$, thus the average L_p-star discrepancy $\mathrm{av}_p^*(N, d)$. Theorem 2.2 gives

$$\lim_{N\to\infty} N^{p/2} \mathrm{av}_p^*(N, d)^p = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int\limits_{[0,1]^d} \left[\prod_{i=1}^d x_i \left(1 - \prod_{i=1}^d x_i\right)\right]^{p/2} \mathrm{d}x,$$

which is the result of Theorem 2.1. For $k > 0$ we compute the integral (2.2.15) and get

$$\int\limits_{[0,1]^d} \left(\prod_{i=1}^d x_i\right)^k \mathrm{d}x = \left(\frac{1}{k+1}\right)^d.$$

Example 2.2 (Quadrant L_p-discrepancy in α)**.** To get $\mathrm{av}_p^\alpha(N, d)$, the average quadrant L_p-discrepancy in $\alpha \in [0, 1]^d$, we choose

$$\Omega_d = [0, 1]^d \text{ and } \mu_d = \lambda^d.$$

The boxes $B(x)$ for fixed $x \in \Omega_d$ are defined as

$$B(x) = \bigtimes_{i=1}^d \left[\chi_{[\alpha_i,1]}(x_i) \cdot x_i, \chi_{[\alpha_i,1]}(x_i) + \chi_{[0,\alpha_i)}(x_i) \cdot x_i\right).$$

These boxes have the Lebesgue measure

$$\lambda^d(B(x)) = \prod_{i=1}^d \left(\chi_{[\alpha_i,1]}(x_i)(1 - x_i) + \chi_{[0,\alpha_i)}(x_i) x_i\right). \tag{2.2.16}$$

Theorem 2.2 gives

$$\lim_{N\to\infty} N^{p/2}\mathrm{av}_p^{\alpha}(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int\limits_{[0,1]^d}\left[\lambda^d(B(x))\Big(1-\lambda^d(B(x))\Big)\right]^{p/2}\mathrm{d}x.$$

with $\lambda^d(B(x))$ given by (2.2.16).

The following picture illustrates the boxes B for different x.

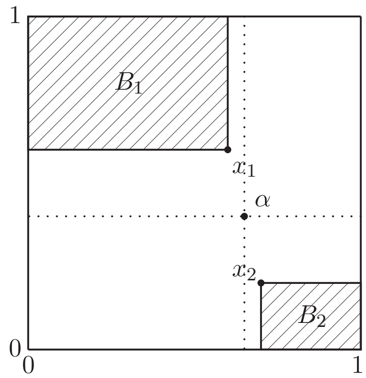

Figure 2.2: Quadrant L_p-discrepancy in $\alpha = [0.65, 0.4]$. Boxes B_1, B_2 for points $x_1, x_2 \in \Omega_2 = [0,1]^2$

For $k > 0$ we compute the integral (2.2.15) and get

$$\begin{aligned}
&\int\limits_{[0,1]^d}\left(\prod_{i=1}^{d}\Big(\chi_{[\alpha_i,1]}(x_i)(1-x_i)+\chi_{[0,\alpha_i)}(x_i)x_i\Big)\right)^k \mathrm{d}x\\
&=\prod_{i=1}^{d}\left(\int\limits_0^{\alpha_i} x_i^k\,\mathrm{d}x_i+\int\limits_{\alpha_i}^{1}(1-x_i)^k\,\mathrm{d}x_i\right)\\
&=\left(\frac{1}{k+1}\right)^d\prod_{i=1}^{d}\Big(\alpha_i^{k+1}+(1-\alpha_i)^{k+1}\Big).
\end{aligned}$$

If we choose $\alpha = (\frac{1}{2},\dots,\frac{1}{2})$ we get the average centered L_p-discrepancy $\mathrm{av}_p^{\boxdot}(N,d)$. Theorem2.2 gives

$$\begin{aligned}
&\lim_{N\to\infty} N^{p/2}\mathrm{av}_p^{\boxdot}(N,d)^p\\
&=\frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int\limits_{[0,1]^d}\left[\prod_{i=1}^{d}\min\{x_i,1-x_i\}\left(1-\prod_{i=1}^{d}\min\{x_i,1-x_i\}\right)\right]^{p/2}\mathrm{d}x.
\end{aligned}$$

For $k > 0$ we compute the integral (2.2.15) and get

$$\int_{[0,1]^d} \left(\prod_{i=1}^{d} \min\{x_i, 1 - x_i\} \right)^k \mathrm{d}x = \left(\frac{1}{2^k} \cdot \frac{1}{k+1} \right)^d .$$

Example 2.3 (Extreme L_p-discrepancy)**.** To get the average extreme L_p-discrepancy $\mathrm{av}_p(N,d)$ on $[0,1]^d$ we choose

$$\Omega_d = \left\{ x = (x_1, x_2) \in [0,1]^d \times [0,1]^d : x_1 \le x_2 \right\} \subset [0,1]^{2d}.$$

The boxes $B(x)$ for fixed $x = (x_1, x_2) \in \Omega_d$ are defined as

$$B(x) = [x_1, x_2).$$

The following picture illustrates the boxes B for different x.

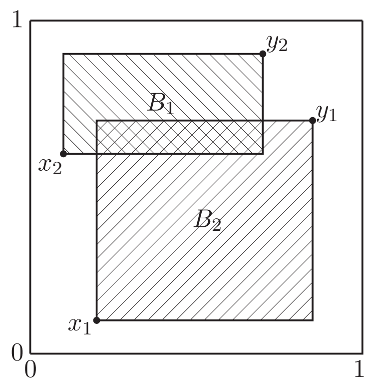

Figure 2.3: Extreme L_p-discrepancy. Boxes B_1, B_2 for points $(x_1, y_1), (x_2, y_2) \in \Omega_2$

The measure μ_d is a normalized Lebesgue measure $c\lambda^{2d}$. To get the normalization factor c, we have to compute $\lambda^{2d}(\Omega_d)$. This yields

$$\lambda^{2d}(\Omega_d) = \int_{[0,1]^d} \int_{[x_1,1]} 1 \,\mathrm{d}x_2 \,\mathrm{d}x_1 = \prod_{i=1}^{d} \left(\int_0^1 \int_{x_1^i}^1 1 \,\mathrm{d}x_2^i \,\mathrm{d}x_1^i \right) = \prod_{i=1}^{d} \left(\int_0^1 (1 - x_1^i) \,\mathrm{d}x_1^i \right) = \left(\frac{1}{2} \right)^d .$$

Hence, we get the measure

$$\mu_d = 2^d \lambda^{2d}.$$

The boxes $B(x)$ for fixed $x = (x_1, x_2) \in \Omega_d$ have the Lebesgue measure

$$\lambda^d(B(x)) = \prod_{i=1}^{d} \left(x_2^i - x_1^i \right).$$

Theorem 2.2 yields

$$\begin{aligned}
&\lim_{N\to\infty} N^{p/2} \mathrm{av}_p(N,d)^p \\
&= \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int\limits_{[0,1]^d} \int\limits_{[x,\mathbf{1}]} \left[\prod_{i=1}^{d} \left(x_2^i - x_1^i \right) \left(1 - \prod_{i=1}^{d} \left(x_2^i - x_1^i \right) \right) \right]^{p/2} 2^d \,\mathrm{d}\mathbf{y}\,\mathrm{d}x.
\end{aligned}$$

For $k > 0$ we compute the integral (2.2.15) and get

$$\begin{aligned}
2^d \int\limits_{[0,1]^d} \int\limits_{[x,\mathbf{1}]} \left(\prod_{i=1}^{d} (y_i - x_i) \right)^k \mathrm{d}\mathbf{y}\,\mathrm{d}x &= 2^d \int\limits_{[0,1]^d} \left(\prod_{i=1}^{d} \int\limits_{x_i}^{1} (y_i - x_i)^k \,\mathrm{d}y_i \right) \mathrm{d}x \\
&= 2^d \int\limits_{[0,1]^d} \left(\prod_{i=1}^{d} \frac{1}{k+1} (1 - x_i)^{k+1} \right) \mathrm{d}x \\
&= 2^d \left(\frac{1}{k+1} \right)^d \left(\prod_{i=1}^{d} \int\limits_{0}^{1} (1 - x_i)^{k+1} \,\mathrm{d}x_i \right) \\
&= \left(\frac{2}{(k+1)(k+2)} \right)^d.
\end{aligned}$$

Example 2.4 (Periodic L_p-discrepancy)**.** To get the average periodic L_p-discrepancy $\mathrm{av}_p^\circ(N,d)$ we choose

$$\Omega_d = [0,1]^d \times [0,1]^d \text{ and } \mu_d = \lambda^{2d}.$$

We define the boxes $B(x)$ for fixed $x = (x_1, x_2) \in \Omega_d$ as

$$B(x) = \bigtimes_{i=1}^{d} \left[x_1^i, \chi_{\{x_1^i > x_2^i\}} + \chi_{\{x_1^i \le x_2^i\}} x_2^i \right) \cup \left[0, \chi_{\{x_1^i > x_2^i\}} x_2^i \right).$$

The following pictures illustrate the boxes B for different x.

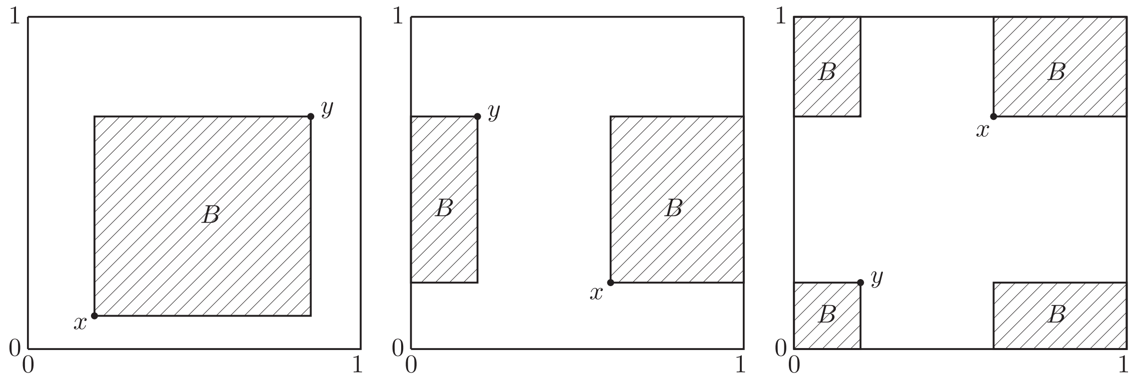

Figure 2.4: Periodic L_p-discrepancy. Boxes B for points $(x, y) \in \Omega_2 = [0,1]^2 \times [0,1]^2$

These boxes have the Lebesgue measure

$$\lambda^d(B(x)) = \prod_{i=1}^{d} \left(\chi_{[0,x_1^i)}(x_2^i)(1 + x_2^i - x_1^i) + \chi_{[x_1^i,1)}(x_2^i)(x_2^i - x_1^i) \right). \tag{2.2.17}$$

Theorem 2.2 gives

$$\lim_{N\to\infty} N^{p/2} \mathrm{av}_p^{\circ}(N,d)^p = \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{1+p}{2}\right) \int_{[0,1]^d} \int_{[0,1]^d} \left[\lambda^d(B(x))\left(1 - \lambda^d(B(x))\right)\right]^{p/2} \mathrm{d}x_1 \,\mathrm{d}x_2,$$

with $\lambda^d(B(x))$ given by (2.2.17).

For $k > 0$ we compute the integral (2.2.15) and get

$$\begin{aligned}
\int_{[0,1]^d} \int_{[0,1]^d} \left(\lambda^d(B(x))\right)^k \mathrm{d}\mathbf{y}\,\mathrm{d}x &= \int_{[0,1]^d} \left(\prod_{i=1}^{d} \left(\int_0^{x_i} (1 + y_i - x_i)^k \,\mathrm{d}y_i + \int_{x_i}^{1} (y_i - x_i)^k \,\mathrm{d}y_i \right) \right) \mathrm{d}x \\
&= \int_{[0,1]^d} \left(\prod_{i=1}^{d} \frac{1}{k+1} \left(1 - (1 - x_i)^{k+1} + (1 - x_i)^{k+1}\right) \right) \mathrm{d}x \\
&= \left(\frac{1}{k+1}\right)^d.
\end{aligned}$$

Example 2.5 (Periodic ball L_p-discrepancy)**.** To define the average periodic ball L_p-discrepancy $\mathrm{av}_p^{\bullet}(N,d)$ let $0 \le r_1 < r_2 < \infty$ and e_j the jth canonical unit vector in dimension d. We choose

$$\Omega_d = [0,1]^d \times [r_1, r_2].$$

The sets $B(\mathbf{y})$ for fixed $\mathbf{y} = (x, r) \in \Omega_d$ are defined as

$$B(x,r) = \bigcup_{J \subset [d]} \left(B_r \left(x + \sum_{j \in J} e_j \right) \cap [0,1]^d \right),$$

where $B_r(x)$ is the open ball around x with radius r.

In the case $p = 2, d = 2$ this type of discrepancy was investigated by Gräf, Potts and Steidel ([27]). The following pictures illustrate the sets B for different x and fixed $r = 1/4$.

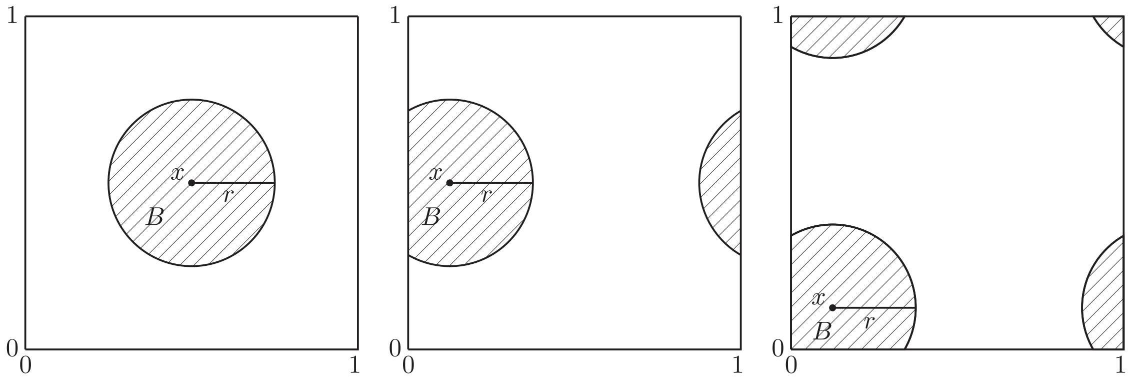

Figure 2.5: Periodic ball L_p-discrepancy. Sets B for $(x,r) \in \Omega_2 = [0,1]^2 \times [0,1/2]$

The measure μ_d is a normalized Lebesgue measure $c\lambda^{d+1}$. To get the normalization factor c, we have to compute $\lambda^{d+1}(\Omega_d)$. This yields

$$\lambda^{d+1}(\Omega_d) = \int_{[0,1]^d} \int_{r_1}^{r_2} 1 \, \mathrm{d}r \, \mathrm{d}x = r_2 - r_1.$$

Hence, we get the measure

$$\mu_d = \frac{1}{r_2 - r_1} \lambda^{d+1}.$$

The sets $B(x,r)$ for fixed $(x,r) \in \Omega_d$ have the Lebesgue measure

$$\lambda^d(B(x,r)) = r^d \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

Theorem 2.2 gives

$$\begin{aligned}
&\lim_{N\to\infty} N^{p/2}\mathrm{av}_p^{\bullet}(N,d)^p \\
&= \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{1+p}{2}\right)\int\limits_{[0,1]^d}\int\limits_{r_1}^{r_2}\left[r^d\frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(1-r^d\frac{\pi^{d/2}}{\Gamma(d/2+1)}\right)\right]^{p/2}\,\mathrm{d}r\,\mathrm{d}x.
\end{aligned}$$

For $k>0$ we compute the integral (2.2.15) and get

$$\begin{aligned}
\int\limits_{[0,1]^d}\int\limits_{r_1}^{r_2}\left(\lambda^d(B(x,r))\right)^k\,\mathrm{d}r\,\mathrm{d}x &= \frac{1}{r_2-r_1}\cdot\frac{\pi^{d/2}}{\Gamma(d/2+1)}\int\limits_{r_1}^{r_2}r^d\,\mathrm{d}r \\
&= \frac{\pi^{d/2}}{\Gamma(d/2+1)}\cdot\frac{r_2^{d+1}-r_1^{d+1}}{(d+1)(r_2-r_1)}.
\end{aligned}$$

3 Estimates for the expected star discrepancy

3.1 Known results about the star discrepancy

A natural question in discrepancy theory is how small the minimal star discrepancy $D^*(N, d)$ can get - and what are point sets which yield a small star discrepancy.

The classical view on this topic is to ask for the behavior in N while the dimension d is fixed. There is a huge theory about constructing point sets with small star discrepancies. Customarily, we speak of so called low-discrepancy point sets $\mathcal{P}$ if their star discrepancy satisfies

$$D^*_\infty(\mathcal{P}) \leq cN^{-1}(\log N)^d \tag{3.1.1}$$

(see [40]). The most famous one-dimensional low-discrepancy point set is the Van der Corput sequence. Furthermore, this sequence is the starting point for many constructions of low-discrepancy point sets. Other famous sets are for example Halton sequences, Hammersley point sets and digital nets (see [16, 21, 40]).

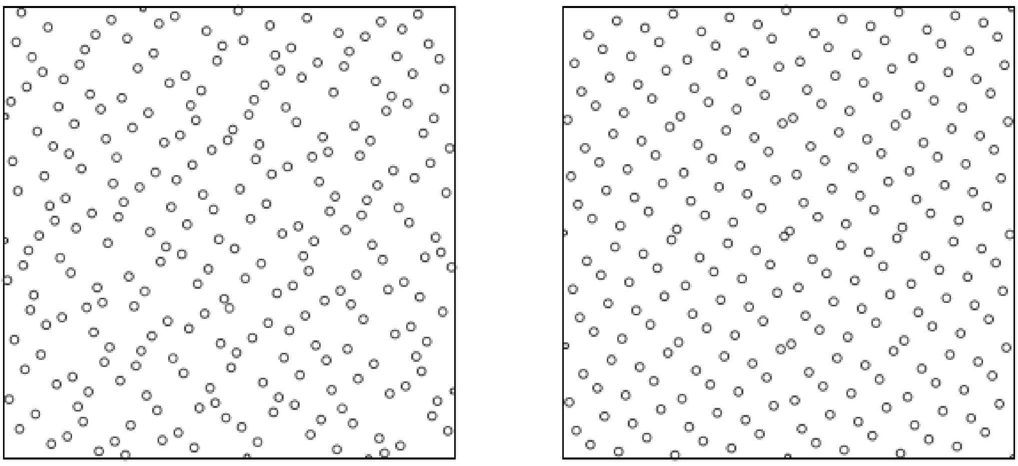

Figure 3.1: Halton and Hammersley point sets

In some literature, a point set is called a low-discrepancy point set only if its star discrepancy has the smaller upper bound $cN^{-1}(\log N)^{d-1}$ (see e.g. [16]).

See [15] for a short survey of many low-discrepancy point sets. It is widely believed that

$$D_\infty^*(\mathcal{P}) \geq c(d)N^{-1}(\log N)^{d-1}$$

holds for any point set $\mathcal{P}$ ([40]), which means $D^*(N,d) \geq c(d)N^{-1}(\log N)^{d-1}$ and therefore that low-discrepancy point sets are of optimal order in N. This conjecture was proved by Schmidt for $d = 2$ ([47]) but is still open for $d \geq 3$. We know directly from Roth's bound for the L_2-discrepancy that

$$D_\infty^*(\mathcal{P}) \geq c(d)N^{-1}(\log N)^{\frac{d-1}{2}}.$$

In 2008, Bilyk, Lacey and Vagharshakyan ([6]) improved that bound to

$$D_\infty^*(\mathcal{P}) \geq c(d)N^{-1}(\log N)^{\frac{d-1}{2}+\eta}$$

for an $\eta = \eta(d) > 0$ with $\eta(d) \to 0$ if $d \to \infty$.

Remarkably, random point sets have expected discrepancy at least of order $N^{-1/2}$. Hence, they are significantly worse than low discrepancy point sets. To show this lower bound let first $N \geq 80$. Let further $B_{3/4} = [0,x] \subset [0,1]^d$ be any box with $\mathrm{vol}(B_{3/4}) = 3/4$. It holds

$$\begin{aligned}
\mathbb{P}\left(D(B_{3/4},\mathcal{P}) \geq \frac{1}{4\sqrt{N}}\right) &= \mathbb{P}\left(ND(B_{3/4},\mathcal{P}) \geq \frac{\sqrt{N/4}}{2}\right) \\
&= \mathbb{P}\left(|\mathcal{P}\cap B_{3/4}| - N\mathrm{vol}(B_{3/4}) \geq \frac{\sqrt{N/4}}{2}\right) \\
&= \mathbb{P}\left(N - \left|\mathcal{P}\cap B_{3/4}^C\right| - N\cdot 3/4 \geq \frac{\sqrt{N/4}}{2}\right).
\end{aligned}$$

Defining now the binomial random variable $X = \sum_{i=1}^N \chi_{B_{3/4}^C}(t_i)$ with parameters N and $p = 1/4$ we can apply Lemma 1.5 and obtain

$$\mathbb{P}\left(D(B_{3/4},\mathcal{P}) \geq \frac{1}{4\sqrt{N}}\right) = \mathbb{P}\left(X \leq N/4 - \frac{\sqrt{N/4}}{2}\right) \geq 3/160.$$

We define

$$B_d = \begin{cases} B_{3/4}, & \text{if } ND(B_{3/4}, \mathcal{P}) \geq \frac{\sqrt{N/4}}{2} \\ B_0 = [0,1]^d, & \text{otherwise.} \end{cases}$$

Hence, these boxes B_d are random variables and therefore, $D(B_d, .)$ is also a random variable with $D(B_d, .) \geq 0$. Applying Markov's inequality (Theorem 1.1) yields

$$\begin{aligned} \mathbb{E}D^*_\infty(\mathcal{P}) &\geq \mathbb{E}D(B_d, \mathcal{P}) \\ &\geq \mathbb{P}\left(D(B_{3/4}, \mathcal{P}) \geq \frac{1}{4\sqrt{N}}\right)\frac{1}{4\sqrt{N}} \\ &\geq 3/640 \cdot \frac{1}{\sqrt{N}}. \end{aligned}$$

Obviously,

$$\mathbb{E}D^*_\infty(\mathcal{P}) \geq c \cdot \frac{1}{\sqrt{N}}$$

holds for all $N < 80$ if c is small enough. Hence, we have for all N the lower bound

$$\mathbb{E}D^*_\infty(\mathcal{P}) \geq c\frac{1}{\sqrt{N}}.$$

For many applications, the dimension d is very large. For example, in finance dimensions d of 360 and higher are not unusual (see [51]). Unfortunately, the function $N^{-1}(\log N)^d$ is increasing for $N \leq e^d$ and therefore, the bound (3.1.1) is of no use for large d. Moreover, the number N of points has to become extremely large to let the term $N^{-1}(\log N)^d$ get smaller then the trivial discrepancy bound 1.

In 1998 Larcher raised the question how $D^*(N,d)$ depends on d (see [30]). More precisely, he asked if $\lim_{d\to\infty} D^*(2^d, d) = 1$ holds. The answer to this question is "No" and was given by Heinrich, Novak, Wasilkowski and Woźniakowski in 2001 ([30]). Among other results they show the upper bound

$$D^*(N,d) \leq c\sqrt{\frac{d}{N}} \tag{3.1.2}$$

with an unknown constant c. The dependence on the dimension d is best possible. The proof of this result is due to the fact that this bound holds for a random point set with positive probability. More precisely the authors show

$$\mathbb{P}\left(D^*_\infty(\mathcal{P}) \geq c\sqrt{\frac{d}{N}}\right) \leq \tilde{c} < 1 \tag{3.1.3}$$

for a random point set $\mathcal{P}$. Aistleitner and Hofer prove (3.1.3) with concrete values for the constant c ([2]). They show for a random point set $\mathcal{P}$ that

$$D_\infty^*(\mathcal{P}) \leq 5.4\sqrt{4.9 + \log\left((1-q)^{-1}\right)}\sqrt{\frac{d}{N}}$$

holds with probability q for any $q \in (0,1)$. Considering the fact that the probability of a random point set satisfying

$$D_\infty^*(\mathcal{P}) \leq c\sqrt{\frac{d}{N}}$$

is very large it is surprising that no general constructions of point sets satisfying such discrepancy bounds are known.

The best known results are a component-by-component-construction of Doerr, Gnewuch, Kritzer and Pillichshammer ([19]), an algorithmic construction via dependent randomized rounding of Doerr, Gnewuch and Wahlström ([20]) and a construction of Hinrichs in dimension $d = 15$, who finds a structured set of $N = 256$ points with discrepancy less than $1/4$ ([33]).

The best known lower bound for the star discrepancy is due to Hinrichs ([32]). He shows

$$D^*(N,d) \geq \min\left\{\varepsilon_0, c\,\frac{d}{N}\right\}$$

with constants $c, \varepsilon_0 > 0$.

Closing the gap between this lower bound and the upper bound of Heinrich, Novak, Wasilkowski and Woźniakowski is a big open problem in discrepancy theory. As Doerr stated in [18], even for random point sets there was no matching lower bound known. To find such a bound he defined the excess of a box B and a point set $\mathcal{P}$ in the following way.

Definition 3.1. Let $B = [0,x] \subset [0,1]^d$ and $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ be a point set. We define the ***excess*** of B and $\mathcal{P}$ by

$$\operatorname{exc}(B) = |\mathcal{P} \cap B| - N\operatorname{vol}(B). \tag{3.1.4}$$

The excess and the star discrepancy are connected via the equality

$$D_\infty^*(\mathcal{P}) = \sup_{x\in[0,1]^d}\left|\frac{1}{N}\operatorname{exc}(B)\right|.$$

Finally, Doerr presents the following theorem.

Theorem 3.1 (Doerr)**.** *There is an absolute constant $K > 0$ such that the following is true:*
Let $N, d \in \mathbb{N}$ with $d \leq N$. Let $\mathcal{P}$ be a set of N points chosen independently and uniformly at random from $[0,1]^d$.
There exists a box $B_d \subset [0,1]^d$ such that the excess satisfies

$$\mathbb{E}\mathrm{exc}(B_d) \geq K \cdot \sqrt{Nd}\,.$$

Hence, the expected star discrepancy satisfies

$$\mathbb{E}(D^*_\infty(\mathcal{P})) \geq K \cdot \sqrt{\frac{d}{N}}\,.$$

The probability that $D^*_\infty(\mathcal{P})$ is less than $K \cdot \sqrt{\frac{d}{N}}$ is at most $\exp(-\Theta(d))$.

3.2 Expectation of weighted star discrepancies

In general, the weights $a_j = 1/N$ are not optimal for the expected star discrepancy. To see this we calculate the optimal weights for the expected weighted star discrepancy in the case $N = d = 1$.

We have

$$D^*_\infty(a, \mathcal{P}) = \sup_{x \in [0,1]^d} |a\chi_B(t) - \mathrm{vol}(B)|\,.$$

If $t \notin B$ we have to maximize $\mathrm{vol}(B)$, which gives

$$\max\{\mathrm{vol}(B) : t \notin B\} = t$$

in dimension $d = 1$. If $t \in B$ we have to maximize $|a - \mathrm{vol}(B)|$, which gives

$$\max\{|a - \mathrm{vol}(B)| : t \in B\} = \max\{a - t, 1 - a\}$$

in dimension $d = 1$. Hence, we have to calculate

$$\inf_{a \geq 0} \mathbb{E}D^*(a, \mathcal{P}) = \inf_{a \geq 0} \int_0^1 \max\{t, a - t, 1 - a\}\, \mathrm{d}t.$$

Now, we have to distinguish four cases:

First case: For $a \in [0, 1/2]$ we have

$$\mathbb{E}D^*_\infty(a, \mathcal{P}) = \int_0^{1-a} (1-a)\ \mathrm{d}t + \int_{1-a}^1 t\,\mathrm{d}t = \frac{a^2}{2} - a + 1$$

and consequently

$$\inf_{a\in[0,1/2]} \mathbb{E}D^*_\infty(a, \mathcal{P}) = \inf_{a\in[0,1/2]} \left(\frac{a^2}{2} - a + 1\right) = \frac{5}{8}.$$

Second case: For $a \in [1/2, 2/3]$ we have

$$\mathbb{E}D^*_\infty(a, \mathcal{P}) = \int_0^{2a-1} (a-t)\ \mathrm{d}t + \int_{2a-1}^{1-a} (1-a)\ \mathrm{d}t + \int_{1-a}^1 t\,\mathrm{d}t = \frac{5}{2}a^2 - 3a + \frac{3}{2}$$

and consequently

$$\inf_{a\in[1/2,2/3]} \mathbb{E}D^*_\infty(a, \mathcal{P}) = \inf_{a\in[1/2,2/3]} \left(\frac{5}{2}a^2 - 3a + \frac{3}{2}\right) = \frac{3}{5}.$$

Third case: For $a \in [2/3, 2]$ we have

$$\mathbb{E}D^*_\infty(a, \mathcal{P}) = \int_0^{a/2} (a-t)\ \mathrm{d}t + \int_{a/2}^1 t\,\mathrm{d}t = \frac{a^2}{4} + \frac{1}{2}$$

and consequently

$$\inf_{a\in[2/3,2]} \mathbb{E}D^*_\infty(a, \mathcal{P}) = \inf_{a\in[2/3,2]} \left(\frac{a^2}{4} + \frac{1}{2}\right) = \frac{11}{18}.$$

Fourth case: For $a \in [2, \infty)$ we have

$$\mathbb{E}D^*_\infty(a, \mathcal{P}) = \int_0^1 (a-t)\ \mathrm{d}t = a - \frac{1}{2}$$

and consequently

$$\inf_{a\in[2,\infty)} \mathbb{E}D^*_\infty(a, \mathcal{P}) = \inf_{a\in[2,\infty)} \left(a - \frac{1}{2}\right) = \frac{3}{2}.$$

Altogether, we have

$$\inf_{a\geq 0} \mathbb{E}D^*(a, \mathcal{P}) = \mathbb{E}D^*_\infty(3/5, \mathcal{P}) = \frac{3}{5} < \frac{3}{4} = \mathbb{E}D^*_\infty(1, \mathcal{P}) = \mathbb{E}D^*_\infty(\mathcal{P}).$$

Hence, it is an interesting question, what happens to the result of Doerr if we allow arbitrary weights and take the infimum over all weights after we calculated the expectation.

To this end, we discuss an adaption of Theorem 3.1. We use Lemma 1.5 as follows: Starting with the box $B_0 = [0,1]^d$ we inductively cut off small boxes of the form $[0,1]^{j-1} \times (1-1/d, 1] \times [0,1]^{d-j}$ for $j \in [d]$ if this increases the absence of points in the rest of the box. Lemma 1.5 gives us, that in each inductive step, we have too many points in the small boxes with positive probability which finally gives us a box B_d with $\Theta(dN)$ points less then expected.

Instead of the excess, which is used by Doerr in [18], we deal with the absence.

Definition 3.2. Let $B = [0,x] \subset [0,1]^d$ and $\mathcal{P} = \{t_1, \ldots, t_N\} \subset [0,1]^d$ be a point set. We define the ***absence*** of B and $\mathcal{P}$ by

$$\mathrm{ab}(B) = N\mathrm{vol}(B) - |\mathcal{P} \cap B| . \tag{3.2.1}$$

Obviously, it holds

$$\mathrm{ab}(B) = -\mathrm{exc}(B)$$

and further

$$D^*_\infty(a, \mathcal{P}) = \sup_{x\in[0,1]^d} \left| \frac{1}{N} \mathrm{ab}(B) \right| .$$

Theorem 3.2. *There is an absolute constant $K > 0$ such that the following is true: Let $N, d \in \mathbb{N}$ with $d \leq N$. Let $\mathcal{P} = \{t_1, \ldots, t_N\}$ be a set of N points chosen independently and uniformly at random from $[0,1]^d$.*
Then there exist boxes $B_d \subset [0,1]^d$, such that the absence satisfies

$$\mathbb{E}\,\mathrm{ab}(B_d) \geq K \cdot \sqrt{Nd}.$$

The following proof is basically the same as the original proof of Doerr for Theorem 3.1. Nevertheless, for better understanding we present it here.

Proof. Since we do not care about the constant K, we may assume $N \geq 80$ for the following proof.

Let $N \geq 80$ and $d < 4$. As in the beginning of this chapter we use again any box $B_{3/4}$ with $\mathrm{vol}(B_{3/4}) = 3/4$ and define a binomial random variable $X = \sum_{i=1}^N \chi_{B^C_{3/4}}(t_i)$ with parameters N and $p = 1/4$. Applying Lemma 1.5 we obtain as before

$$\mathbb{P}\left(\mathrm{ab}(B_{3/4}) \geq \frac{\sqrt{N/4}}{2} \right) = \mathbb{P}\left(X \geq N/4 + \frac{\sqrt{N/4}}{2} \right) \geq 3/160.$$

We define

$$B_d = \begin{cases} B_{3/4}, & \text{if } \operatorname{ab}(B_{3/4}) \geq \frac{\sqrt{N/4}}{2} \\ B_0 = [0,1]^d, & \text{otherwise.} \end{cases}$$

Hence, these boxes B_d are random variables and therefore, $\operatorname{ab}(B_d)$ is also a random variable with $\operatorname{ab}(B_d) \geq 0$. This leads to

$$\begin{aligned} \mathbb{E}\operatorname{ab}(B_d) &\geq \mathbb{P}\left(\operatorname{ab}(B_{3/4}) \geq \frac{\sqrt{N/4}}{2}\right)\frac{\sqrt{N/4}}{2} \\ &\geq 3/160 \cdot \frac{\sqrt{N/4}}{2} \\ &= 3/640 \cdot \sqrt{N} \\ &\geq 1/640\sqrt{Nd}. \end{aligned}$$

Let now $4 \leq d \leq N/4$. We inductively define numbers $x_1, \ldots, x_d \in \{1 - 1/d, 1\}$ such that for fixed $i \in [d]$ the box

$$B_i = \bigtimes_{j=1}^{i} [0, x_j] \bigtimes [0,1]^{d-i}$$

has expected absence of order $i\sqrt{\frac{N}{d}}$.

We have

$$\operatorname{vol}(B_i) \geq (1 - 1/d)^d \geq 1/4. \tag{3.2.2}$$

To see this, observe that

$$(1 - 1/d)^d \geq 1/4 \iff 1 - 1/d \geq 4^{-1/d} \iff 1 \geq 4^{-1/d} + 1/d =: f(d).$$

We have $f(4) = 1/4 + 1/\sqrt{2} < 0.25 + 0.70711 < 1$ and $\lim_{d\to\infty} f(d) = 1$. Hence, it is enough to show that f has exactly one extremal point and this point is a minimum. Because f is continuous as a function in $d \in \mathbb{R}$ we get $f(d) \leq 1$ for all $d \geq 4$. The condition above mentioned is fulfilled because

$$f'(d) = 1/d^2 \left(4^{-1/d} \ln 4 - 1\right) = 0 \iff 4^{1/d} = \ln 4 \iff d = \frac{\ln 4}{\ln(\ln 4)} \approx 4.244 > 4$$

and

$$f\left(\frac{\ln 4}{\ln(\ln 4)}\right) \approx 0.9558 < f(4).$$

Now, we define the boxes B_i. First, we define $B_0 = [0,1]^d$ and get

$$\mathrm{ab}(B_0) = N \cdot 1 - N = 0.$$

Assume now, that for fixed $0 \leq i < d$ the numbers $x_1, \dots, x_i$ are fixed and consequently $B_1, \dots, B_i$ are fixed. Now we define the set of indices $I = \{j \in [N] : t_j \in B_i\}$, the point set $\mathcal{P}_i = \mathcal{P} \cap B_i = \{t_j \in \mathcal{P} : t_j \in B_i\}$ and the number $N_i = |\mathcal{P}_i| = |I|$. Hence, we get for the absence of B_i

$$\mathrm{ab}(B_i) = N\mathrm{vol}(B_i) - |\mathcal{P} \cap B_i| = N\mathrm{vol}(B_i) - N_i.$$

Let C_{i+1} be a subset of B_i defined as

$$C_{i+1} = \bigtimes_{j=1}^{i} [0, x_j] \times (1 - 1/d, 1] \times [0,1]^{d-i-1}.$$

Because the $(i+1)$st to dth coordinates of t_j are independent and uniformly distributed in $[0,1]$ we have for all $j \in I$

$$\mathbb{P}\left(t_j \in C_{i+1}\right) = \mathrm{vol}\left((1 - 1/d, 1]\right) = 1/d.$$

For all $j \in I$ we define random variables

$$X_j^i = \chi_{C_{i+1}}(t_j)$$

and further the sum of these random variables as

$$X^i = \sum_{j \in I} X_j^i = \sum_{j \in I} \chi_{C_{i+1}}(t_j).$$

Obviously, X^i is a binomial distributed random variable with parameters N_i and $p = 1/d$.

Let us now assume that we have $N_i \geq N/5 \geq 16$. Then we can apply Lemma 1.5 and get

$$\mathbb{P}\left(X^i \geq N_i/d + \frac{\sqrt{N_i/d}}{2}\right) \geq \frac{3}{160}.$$

First case: $X^i < N_i/d + \sqrt{N_i/d}/2$.
We choose $x_{i+1} = 1$. Hence, we get $B_{i+1} = B_i$ and therefore $\mathrm{ab}(B_{i+1}) = \mathrm{ab}(B_i)$.
Second case: $X^i \geq N_i/d + \sqrt{N_i/d}/2$.
In this case we choose $x_{i+1} = 1 - 1/d$ and get $B_{i+1} = B_i \setminus C_{i+1}$. This yields for the

absence of B_{i+1}

$$\begin{aligned}
\mathrm{ab}(B_{i+1}) &= N\mathrm{vol}(B_i \setminus C_{i+1}) - |\mathcal{P} \cap (B_i \setminus C_{i+1})| \\
&= N\mathrm{vol}(B_i)\,(1-1/d) - |\mathcal{P} \cap B_i| + |P \cap C_{i+1}| \\
&\geq N\mathrm{vol}(B_i)\,(1-1/d) - N_i + N_i/d + \frac{\sqrt{N_i/d}}{2} \\
&= (1-1/d)\ \mathrm{ab}(B_i) + \frac{\sqrt{N_i/d}}{2} \\
&\geq (1-1/d)\ \mathrm{ab}(B_i) + \frac{\sqrt{N/d}}{\sqrt{5}\cdot 2}.
\end{aligned} \tag{3.2.3}$$

Let k be the number of x_i with $x_i = 1-1/d$. Combining (3.2.3) and (3.2.2) leads to

$$\mathrm{ab}(B_d) \geq k(1-1/d)^k \frac{\sqrt{N/d}}{\sqrt{5}\cdot 2} \geq k\sqrt{N/d}\cdot\frac{1}{\sqrt{5}\cdot 8}. \tag{3.2.4}$$

The first inequality is proved by induction over k:
For $k=0$ we have $\mathrm{ab}(B_d) = \mathrm{ab}(B_0) = 0 \geq 0\cdot(1-1/d)^0\frac{\sqrt{N/d}}{\sqrt{5}\cdot 2}$.
Assume now, the inequality is true for all numbers less or equal than k and we want to show it for $k+1$. Assume that the index j is the biggest number with $0 \leq j < d$ and $x_{j+1} = 1-1/d$. Then

$$\begin{aligned}
\mathrm{ab}(B_d) &= \mathrm{ab}(B_{j+1}) \\
&\geq (1-1/d)\,\mathrm{ab}(B_j) + \frac{\sqrt{N/d}}{\sqrt{5}\cdot 2} \\
&\geq (1-1/d)k(1-1/d)^k\frac{\sqrt{N/d}}{\sqrt{5}\cdot 2} + (1-1/d)^{k+1}\frac{\sqrt{N/d}}{\sqrt{5}\cdot 2} \\
&= (k+1)(1-1/d)^{k+1}\frac{\sqrt{N/d}}{\sqrt{5}\cdot 2}.
\end{aligned}$$

In the case $N_i < N/5$ for at least one $0 \leq i < d$ we choose $B_d = [0, 1-1/d]^d$ and get because of $|\mathcal{P} \cap B_d| \leq N_i < N/5$ and (3.2.2) the estimate

$$\begin{aligned}
\mathrm{ab}(B_d) &= N\mathrm{vol}(B_d) - |\mathcal{P} \cap B_d| \\
&\geq N\cdot 1/4 - N/5 \\
&= N/20.
\end{aligned}$$

Obviously, we have for the number k the estimates $0 \le k \le d$ which gives

$$\mathrm{ab}(B_d) \ge k\sqrt{N/d} \cdot \frac{1}{\sqrt{5} \cdot 8}.$$

This is true because

$$k\sqrt{N/d} \le d\sqrt{N/d} = \sqrt{Nd} \le \sqrt{N^2/4} = N/2 \le 10\,\mathrm{ab}(B_d) \le \sqrt{5} \cdot 8\,\mathrm{ab}(B_d).$$

Hence, we have for all point sets $\mathcal{P}$ the estimate

$$\mathbb{E}\,\mathrm{ab}(B_d) \ge \sqrt{N/d} \cdot \frac{1}{\sqrt{5} \cdot 8}\mathbb{E}k. \tag{3.2.5}$$

We describe now k as a sum of random variables, namely $k = \sum_{i=1}^d k_i$ with random variables k_i defined as

$$k_i = \begin{cases} 1, & \text{if } x_i = 1 - 1/d \text{ and } N_j \ge N/5 \ \ \forall j \in [N], \\ 0, & \text{otherwise.} \end{cases}$$

We have for the expectation of k the equality

$$\mathbb{E}k = \sum_{i=1}^d \mathbb{E}k_i = \sum_{i=1}^d \mathbb{P}(k_i = 1). \tag{3.2.6}$$

To estimate this probability we use again (3.2.2) and get for any $B = [0, x] \subset [0, 1]^d$ with $\mathrm{vol}(B) = 1/4$

$$\begin{aligned} \mathbb{P}\left(N_j \ge N/5 \text{ for all } j\right) &\ge \mathbb{P}\left(\left|\mathcal{P} \cap [0, 1 - 1/d]^d\right| \ge N/5\right) \\ &\ge \mathbb{P}\left(|\mathcal{P} \cap B| \ge N/5\right) \\ &\ge \mathbb{P}\left(|\mathcal{P} \cap B| \ge N/4\right) \\ &\ge 1/2. \end{aligned}$$

Applying Lemma 1.5 leads to

$$\begin{aligned} \mathbb{P}\left(k_i = 1\right) &= \mathbb{P}\left(N_j \ge N/5 \ \ \forall j \in [N]\right) \cdot \mathbb{P}\left(x_i = 1 - \frac{1}{d}\middle| N_j \ge N/5 \ \ \forall j \in [N]\right) \\ &\ge 1/2 \cdot \mathbb{P}\left(x_i = 1 - \frac{1}{d}\middle| N_j \ge N/5 \ \ \forall j \in [N]\right) \\ &\ge 1/2 \cdot 3/160 \\ &= 3/320. \end{aligned} \tag{3.2.7}$$

Using (3.2.5), (3.2.6) and (3.2.7) we finally get

$$\mathbb{E}(\operatorname{ab}(B_d)) \geq \sqrt{N/d} \cdot \frac{1}{\sqrt{5} \cdot 8} \mathbb{E}k \geq \sqrt{N/d} \cdot \frac{1}{\sqrt{5} \cdot 8} \cdot \frac{3}{320} d = \sqrt{Nd} \frac{3}{\sqrt{5} \cdot 2560}. \quad (3.2.8)$$

Let now $N/4 < d \leq N$. In this case we project $\mathcal{P}$ onto its first $d' = \lfloor N/4 \rfloor$ coordinates and apply the above case for $4 \leq d' \leq N/4$. This means, we find a box $B' \subset [0,1]^{d'}$ with large absence and define $B = B' \times [0,1]^{d-d'}$. This gives

$$\operatorname{vol}(B) = \operatorname{vol}(B') \text{ and } \chi_B(t_j) = \chi_{B'}(t_j^{\perp}).$$

Together with

$$d' \geq \frac{N-3}{4} \geq \frac{d}{4} - \frac{3d}{N} \geq d\left(\frac{1}{4} - \frac{3}{80}\right) = \frac{17}{80} d$$

we obtain

$$\mathbb{E}(\operatorname{ab}(B)) = \mathbb{E}(\operatorname{ab}(B')) \geq \frac{\sqrt{Nd'} \cdot 3}{2560 \cdot \sqrt{5}} \geq \sqrt{N \frac{17d}{80}} \cdot \frac{3}{2560 \cdot \sqrt{5}} = \sqrt{Nd} \cdot \frac{3 \cdot \sqrt{17}}{51200}.$$

$\square$

To show the lower bound for the expected weighted star discrepancy with arbitrary weights, we need the following lemma.

Lemma 3.1. *Let $a = (a_j)_{j=1}^N$ be a sequence of weights and let $c = (c_j)_{j=1}^N$ be another sequence of weights, defined as $c_i = 1/N \cdot \sum_{j=1}^N a_j$. Then the following inequality holds*

$$\mathbb{E}D_\infty^*(a, \mathcal{P}) \geq \mathbb{E}D^*(c, \mathcal{P}). \quad (3.2.9)$$

Proof. For $i, j \in [N]$ we define permutations σ_j as

$$\sigma_j(a_i) = \begin{cases} a_{i+j}, & \text{if } i+j \leq N, \\ a_{i+j-N}, & \text{if } i+j > N. \end{cases}$$

We have for each $i \in [N]$ and each $j \in [N]$ the equation

$$\sum_{j=1}^N \sigma_j(a_i) = \sum_{i=1}^N \sigma_j(a_i) = \sum_{j=1}^N a_j.$$

Hence, we have for the weights c_i

$$c_i = \frac{1}{N} \sum_{j=1}^{N} a_j = \frac{1}{N} \sum_{j=1}^{N} \sigma_j(a_i).$$

This yields

$$\begin{aligned}
ND(c, B, \mathcal{P}) &= N \left[\sum_{i=1}^{N} c_i \chi_B(t_i) - \mathrm{vol}(B) \right] \\
&= N \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} \sigma_j(a_i) \chi_B(t_i) - \mathrm{vol}(B) \right] \\
&= \sum_{j=1}^{N} \left[\sum_{i=1}^{N} \sigma_j(a_i) \chi_B(t_i) - \mathrm{vol}(B) \right] \\
&= \sum_{j=1}^{N} \left[D(\sigma_j(a), B, \mathcal{P}) \right].
\end{aligned}$$

Applying the triangle inequality gives for each point set $\mathcal{P}$ of N independent and uniformly distributed points in $[0,1]^d$

$$\begin{aligned}
N \mathbb{E}_{\mathcal{P}} D_\infty^*(c, \mathcal{P}) &= \mathbb{E}_{\mathcal{P}} \sup_{x \in [0,1]^d} N \left| D(c, B, \mathcal{P}) \right| \\
&= \mathbb{E}_{\mathcal{P}} \sup_{x \in [0,1]^d} \left| \sum_{j=1}^{N} D(\sigma_j(a), B, \mathcal{P}) \right| \\
&\leq \mathbb{E}_{\mathcal{P}} \sup_{x \in [0,1]^d} \sum_{j=1}^{N} \left| D(\sigma_j(a), B, \mathcal{P}) \right| \\
&\leq \sum_{j=1}^{N} \mathbb{E}_{\mathcal{P}} \sup_{x \in [0,1]^d} \left| D(\sigma_j(a), B, \mathcal{P}) \right| \\
&= \sum_{j=1}^{N} \mathbb{E}_{\mathcal{P}} D_\infty^*(\sigma_j(a), \mathcal{P}).
\end{aligned}$$

The proof is complete if we show

$$\mathbb{E}_{\mathcal{P}} D_\infty^*(\sigma_j(a), \mathcal{P}) = \mathbb{E}_{\mathcal{P}} D_\infty^*(a, \mathcal{P}).$$

To this end, we apply the permutations σ_j also to the point set $\mathcal{P}$, which gives

$$\sigma_j(t_i) = \begin{cases} t_{i+j}, & \text{if } i+j \leq N, \\ t_{i+j-N}, & \text{if } i+j > N. \end{cases}$$

Because $\sigma_j(\mathcal{P})$ is a permutation of the point set $\mathcal{P}$ and, therefore, also a set of N independent and uniformly distributed points in $[0,1]^d$, we get

$$\begin{aligned} \mathbb{E}_{\mathcal{P}} D^*_\infty(\sigma_j(a), \mathcal{P}) &= \mathbb{E}_{\sigma_j(\mathcal{P})} D^*_\infty(\sigma_j(a), \sigma_j(\mathcal{P})) \\ &= \mathbb{E}_{\mathcal{P}} D^*_\infty(\sigma_j(a), \sigma_j(\mathcal{P})). \end{aligned}$$

Now, both the point set and the weights in the discrepancy function are permuted with σ_j. Because the order of the summands $a_i \chi_B(t_i)$ in the discrepancy function is irrelevant, we get

$$\mathbb{E}_{\mathcal{P}} D^*_\infty(\sigma_j(a), \mathcal{P}) = \mathbb{E}_{\mathcal{P}} D^*_\infty(\sigma_j(a), \sigma_j(\mathcal{P})) = \mathbb{E}_{\mathcal{P}} D^*_\infty(a, \mathcal{P}).$$

□

Theorem 3.3. *There is an absolute constant $K > 0$ such that the following is true: Let $N, d \in \mathbb{N}$ with $d \leq N$. Let $\mathcal{P}$ be a set of N points chosen independently and uniformly at random from $[0,1]^d$. Let further $a = (a_j)_{j=1}^N$ be a sequence of weights. Then the expected weighted star discrepancy satisfies*

$$\mathbb{E} D^*_\infty(a, \mathcal{P}) \geq K \cdot \sqrt{\frac{d}{N}}.$$

Proof. **First case**: Let $\sum_{j=1}^N a_j = 1$. We apply Lemma 3.1 with weights $c_i = 1/N$ and get

$$\mathbb{E} D^*_\infty(a, \mathcal{P}) \geq \mathbb{E} D^*_\infty(c, \mathcal{P}) = \mathbb{E} D^*_\infty(\mathcal{P}).$$

Now, we use Theorem 3.1 and get

$$\mathbb{E} D^*_\infty(a, \mathcal{P}) \geq \mathbb{E} D^*_\infty(\mathcal{P}) \geq K \sqrt{\frac{d}{N}}.$$

Second case: Let $\sum_{j=1}^N a_j \geq 1$. We apply again Lemma 3.1. For the weights we now have $c_i = 1/N \cdot \sum_{j=1}^N a_j \geq 1/N$. Hence, we get

$$\mathbb{E} D^*_\infty(a, \mathcal{P}) \geq \mathbb{E} D^*(c, \mathcal{P}). \tag{3.2.10}$$

Further, we have for all boxes $B = [0,x] \subset [0,1]^d$ and all point sets $\mathcal{P}$ of N independent and uniformly distributed points in $[0,1]^d$

$$D(c,B,\mathcal{P}) = \sum_{j=1}^{N} c_j \chi_B(t_j) - \operatorname{vol}(B) \geq \sum_{j=1}^{N} \frac{1}{N} \chi_B(t_j) - \operatorname{vol}(B) = D(B,\mathcal{P}). \quad (3.2.11)$$

For a fixed point set $\mathcal{P}$ let B_d be the box for which Theorem 3.1 gives

$$\mathbb{E} D(B_d,\mathcal{P}) \geq K\sqrt{\frac{d}{N}}\,. \quad (3.2.12)$$

Finally, we get

$$\begin{aligned}
\mathbb{E} D_\infty^*(a,\mathcal{P}) &\overset{(3.2.10)}{\geq} \mathbb{E} D^*(c,\mathcal{P}) \\
&\geq \mathbb{E} D(c,B_d,\mathcal{P}) \\
&\overset{(3.2.11)}{\geq} \mathbb{E} D(B_d,\mathcal{P}) \\
&\overset{(3.2.12)}{\geq} K\sqrt{\frac{d}{N}}\,.
\end{aligned}$$

Third case: Let $\sum_{j=1}^N a_j \leq 1$. We apply again Lemma 3.1. For the weights we now have $c_i = 1/N \cdot \sum_{j=1}^N a_j \leq 1/N$. Hence, we get

$$\mathbb{E} D_\infty^*(a,\mathcal{P}) \geq \mathbb{E} D^*(c,\mathcal{P}). \quad (3.2.13)$$

Further, we have for all boxes $B = [0,x] \subset [0,1]^d$ and all point sets $\mathcal{P}$ of N independent and uniformly distributed points in $[0,1]^d$

$$-D(c,B,\mathcal{P}) = \operatorname{vol}(B) - \sum_{j=1}^{N} c_j \chi_B(t_j) \geq \operatorname{vol}(B) - \sum_{j=1}^{N} \frac{1}{N} \chi_B(t_j) = -D(B,\mathcal{P}). \quad (3.2.14)$$

For a fixed point set $\mathcal{P}$ let B_d be the box for which Theorem 3.2 gives

$$\mathbb{E}(-D(B_d,\mathcal{P})) = 1/N \cdot \mathbb{E}\,\operatorname{ab}(B_d) \geq K\sqrt{\frac{d}{N}}\,. \quad (3.2.15)$$

Finally, we get

$$\begin{aligned}
\mathbb{E}D^*_\infty(a,\mathcal{P}) &\overset{(3.2.13)}{\geq} \mathbb{E}D^*(c,\mathcal{P}) \\
&\geq \mathbb{E}(-D(c,B_d,\mathcal{P})) \\
&\overset{(3.2.14)}{\geq} \mathbb{E}(-D(B_d,\mathcal{P})) \\
&\overset{(3.2.15)}{\geq} K\sqrt{\frac{d}{N}}\,.
\end{aligned}$$

□

3.3 Expectation of the optimally weighted star discrepancy for d=1

It is an obvious question how $\mathbb{E}D^*_\infty(a,\mathcal{P})$ behaves if we use not arbitrary, fixed weights but the best possible weights for the point set $\mathcal{P}=\{t_1,\dots,t_N\}$.

For $d=1$, we want to find bounds for

$$\mathbb{E}\inf_a D^*_\infty(a,\mathcal{P})$$

In this case, we assume without loss of generality $0\leq t_1\leq\cdots\leq t_N\leq 1$.

First, we want to find a formula for the weighted star discrepancy.

Lemma 3.2. *For the weighted star discrepancy in dimension $d=1$ holds*

$$D^*_\infty(a,\mathcal{P}) = \max\left\{\sum_{j=1}^{i} a_j - t_i, t_i - \sum_{j=1}^{i-1} a_j, 1-\sum_{j=1}^{N} a_j \,:\, i=1,\dots,N\right\}. \tag{3.3.1}$$

Proof. The star discrepancy is defined as

$$D^*_\infty(a,\mathcal{P}) = \sup_{x\in[0,1]^d} |D(a,B,\mathcal{P})| = \sup_{x\in[0,1]^d}\left|\sum_{j=1}^{N} a_j\chi_B(t_j) - \mathrm{vol}(B)\right|.$$

To compute this, we have to find optimal boxes B. This means, we need to find boxes B which maximize $|D(a,B,\mathcal{P})|$ for a fixed point set $\mathcal{P}$ and fixed weights a.

To this end, we discuss two possible cases. If

$$\left|\sum_{j=1}^{N} a_j \chi_B(t_j) - \mathrm{vol}(B)\right| = \sum_{j=1}^{N} a_j \chi_B(t_j) - \mathrm{vol}(B),$$

the optimal box is one of the boxes $B = [0, t_i]$ with $i = 1, \ldots, N$. This gives

$$D(a, B, \mathcal{P}) = \sum_{j=1}^{i} a_j - t_i.$$

If

$$\left|\sum_{j=1}^{N} a_j \chi_B(t_j) - \mathrm{vol}(B)\right| = -\sum_{j=1}^{N} a_j \chi_B(t_j) + \mathrm{vol}(B),$$

the optimal box is one of the boxes $B = [0, t_i - \varepsilon]$ with $i = 1, \ldots, N$ and $\varepsilon > 0$ very small. This gives

$$D(a, B, \mathcal{P}) = t_i - \sum_{j=1}^{i-1} a_j$$

if $\varepsilon \to 0$. There is one special case left, namely $B = [0, 1]$ which gives

$$D(a, B, \mathcal{P}) = 1 - \sum_{j=1}^{N} a_j.$$

This three cases together yield the lemma. □

3.3.1 Upper bounds

First, we prove a probabilistic lemma.

Lemma 3.3. *Let $m \in [N]$ and let $I_j^m = [\frac{j-1}{m}, \frac{j}{m}]$ for $j = 1, \ldots, m$. We denote by E the following event: There is at least one interval I_j^m, where none of the points $t_1, \ldots, t_N$ lies in. It holds*

$$\mathbb{P}(E) \le m \left(1 - \frac{1}{m}\right)^N. \tag{3.3.2}$$

Proof. We want to estimate the probability for the event E. By E_j we denote the event that no point lies in I_j^m. Because the points $t_1, \ldots, t_N$ are uniformly distributed, we have

$$\mathbb{P}\left(t_i \notin I_j^m\right) = 1 - \frac{1}{m}$$

for all $i = 1, \dots, N$. Because $t_1, \dots, t_N$ are independent, we have

$$\mathbb{P}(E_j) = \left(1 - \frac{1}{m}\right)^N.$$

Now, we use $E = \bigcup_{j=1}^m E_j$ and the union bound and get

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_{j=1}^m E_j\right) \le \sum_{j=1}^m \mathbb{P}(E_j) = m\left(1 - \frac{1}{m}\right)^N.$$

□

From this lemma we easily get:

Lemma 3.4. *Let $m \in [N]$. If the event E^C takes place, we have*

$$\inf_a D_\infty^*(a, \mathcal{P}) \le \frac{1}{m}.$$

Hence, we have because of Lemma 3.3

$$\mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \le \frac{1}{m}\right) \ge \mathbb{P}\left(E^C\right) \ge 1 - m\left(1 - \frac{1}{m}\right)^N.$$

Proof. If the event E^C takes place, there is at least one point in each interval I_j^m. We choose now exactly one point out of each interval I_j^m. We denote these points by $\widetilde{\mathcal{P}} = \left\{\tilde{t}_1, \dots, \tilde{t}_m\right\}$, where $\tilde{t}_j \in I_j^m$. Now, we take the weights $\widetilde{a}_j = \frac{1}{m}$ if $t_j \in \widetilde{\mathcal{P}}$ and $\widetilde{a}_j = 0$ otherwise. Applying Lemma 3.2 we get

$$\begin{aligned}
\inf_a D_\infty^*(a, \mathcal{P}) &\le D_\infty^*(\widetilde{a}, \mathcal{P}) \\
&= D^*\left(\frac{1}{m}, \widetilde{\mathcal{P}}\right) \\
&= \sup_{x \in [0,1]^d} \left| \frac{1}{m} \sum_{j=1}^m \chi_B(\tilde{t}_j) - \mathrm{vol}(B) \right| \\
&= \max\left\{\tilde{t}_j - \frac{j-1}{m}, \frac{j}{m} - \tilde{t}_j \,:\, j = 1, \dots, m\right\} \\
&\le \max\left\{\frac{j}{m} - \frac{j-1}{m}, \frac{j}{m} - \frac{j-1}{m} \,:\, j = 1, \dots, m\right\} \\
&= \frac{1}{m}.
\end{aligned}$$

□

Theorem 3.4. *Let $m \in [N]$. Then we have for the expectation of the optimally weighted star discrepancy in dimension $d = 1$ the upper bound*

$$\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) \leq m \left(1 - \frac{1}{m}\right)^N + \frac{1}{m}.$$

Proof. We have for the expectation

$$\begin{aligned}\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) &= \int_\Omega \inf_a D^*_\infty(a, \mathcal{P}) \,\mathrm{d}\mathbb{P} \\ &= \int_E \inf_a D^*_\infty(a, \mathcal{P}) \,\mathrm{d}\mathbb{P} + \int_{E^C} \inf_a D^*_\infty(a, \mathcal{P}) \,\mathrm{d}\mathbb{P} \\ &\leq \int_E 1 \,\mathrm{d}\mathbb{P} + \int_{E^C} \inf_a D^*_\infty(a, \mathcal{P}) \,\mathrm{d}\mathbb{P}.\end{aligned}$$

Now we apply Lemma 3.3 and Lemma 3.4 and get

$$\begin{aligned}\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) &\leq \mathbb{P}(E) \cdot 1 + \mathbb{P}\left(E^C\right) \cdot \frac{1}{m} \\ &\leq m \left(1 - \frac{1}{m}\right)^N \cdot 1 + 1 \cdot \frac{1}{m}.\end{aligned}$$

□

Upper bounds with decay N^α

Corollary 3.1. *Let $\alpha \in [1/2, 1)$. Then we have for the expectation of the optimally weighted star discrepancy in dimension $d = 1$ the upper bound*

$$\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) \leq C(\alpha) \cdot \frac{1}{N^\alpha},$$

with

$$\lim_{\alpha \to 1^-} C(\alpha) = \infty.$$

Proof. We apply Theorem 3.4. Hence, we have

$$\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) \leq m \left(1 - \frac{1}{m}\right)^N + \frac{1}{m}. \tag{3.3.3}$$

We know that

$$m \left(1 - \frac{1}{m}\right)^N = m \left(\left(1 - \frac{1}{m}\right)^m\right)^{\frac{N}{m}} \leq m \left(\frac{1}{e}\right)^{\frac{N}{m}}. \tag{3.3.4}$$

We choose $m = \lfloor N^\alpha \rfloor$ and show

$$m\left(\frac{1}{e}\right)^{\frac{N}{m}} \leq c \cdot \frac{1}{m}. \tag{3.3.5}$$

This is equivalent to

$$m^2 \leq c \cdot e^{N/m}.$$

Because

$$m^2 = \lfloor N^\alpha \rfloor^2 \leq N^{2\alpha}$$

and

$$e^{N/m} = e^{\frac{N}{\lfloor N^\alpha \rfloor}} \geq e^{\frac{N}{N^\alpha}} = e^{N^{1-\alpha}}$$

it is enough to show

$$N^{2\alpha} \leq c \cdot e^{N^{1-\alpha}}.$$

But this is true for all N if we choose $c = c(\alpha)$ big enough. Observe, that $c(\alpha)$ tends to infinity if α tends to 1. Inequalities (3.3.3), (3.3.4) and (3.3.5) together yield

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \leq (c(\alpha)+1)\frac{1}{\lfloor N^\alpha \rfloor} \leq 2(c(\alpha)+1)\frac{1}{N^\alpha},$$

if $N \geq 4$. This completes the proof for $N \geq 4$. If $N < 4$ Corollary 3.1 is obviously true if $C(\alpha) \geq 3$. $\square$

Upper bounds with faster decay

Corollary 3.2. *Let $N \geq 2$. Then it holds*

$$\mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \leq 2 \cdot \frac{\log N}{N}\right) \geq 0.082. \tag{3.3.6}$$

Proof. If $N \geq 3$ we choose $m = \lfloor N/\log N \rfloor$ in inequality (3.3.4) and get

$$\left\lfloor \frac{N}{\log N} \right\rfloor \left(1 - \frac{1}{\left\lfloor \frac{N}{\log N} \right\rfloor}\right)^N \leq \left\lfloor \frac{N}{\log N} \right\rfloor \left(\frac{1}{e}\right)^{\frac{N}{\left\lfloor \frac{N}{\log N} \right\rfloor}}. \tag{3.3.7}$$

We now estimate

$$\begin{aligned}\left\lfloor \frac{N}{\log N} \right\rfloor \left(1 - \frac{1}{\left\lfloor \frac{N}{\log N} \right\rfloor}\right)^N &\le \frac{N}{\log N} \left(\frac{1}{e}\right)^{\frac{N}{\frac{N}{\log N}}} \\ &= \frac{N}{\log N} \left(\frac{1}{e}\right)^{\log N} \\ &= \frac{N}{\log N} \cdot \frac{1}{N} \\ &= \frac{1}{\log N} \\ &< \frac{1}{1.09} \\ &< 0.918. \end{aligned} \tag{3.3.8}$$

For all $N \ge 2$ holds

$$\left\lfloor \frac{N}{\log N} \right\rfloor \ge \frac{N}{\log N} - 1 = \frac{N - \log N}{\log N} \ge \frac{N - N/2}{\log N} = \frac{1}{2} \cdot \frac{N}{\log N}. \tag{3.3.9}$$

Inequality (3.3.9), Lemma 3.4 and inequality (3.3.8) yield

$$\begin{aligned}\mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \le 2 \cdot \frac{\log N}{N}\right) &\ge \mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \le \frac{1}{\left\lfloor \frac{N}{\log N} \right\rfloor}\right) \\ &\ge 1 - \left\lfloor \frac{N}{\log N} \right\rfloor \left(1 - \frac{1}{\left\lfloor \frac{N}{\log N} \right\rfloor}\right)^N \\ &\ge 0.082. \end{aligned}$$

If $N = 2$ we apply Lemma 3.4 and get

$$\mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \le 2 \cdot \frac{\log(2)}{2}\right) \ge \mathbb{P}\left(\inf_a D_\infty^*(a, \mathcal{P}) \le \frac{1}{2}\right) \ge 1 - 2\left(1 - \frac{1}{2}\right)^2 \ge 0.082.$$

□

Corollary 3.3. *We have for the expectation of the optimally weighted star discrepancy in dimension $d = 1$ the upper bound*

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \le 8 \cdot \frac{\log N}{N}.$$

Proof. We apply Theorem 3.4. Hence, we have

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \leq m \left(1 - \frac{1}{m}\right)^N + \frac{1}{m} \tag{3.3.10}$$

with $m = \left\lfloor \frac{1}{2} \cdot \frac{N}{\log N} \right\rfloor$. We know that

$$m \left(1 - \frac{1}{m}\right)^N = m \left(\left(1 - \frac{1}{m}\right)^m\right)^{\frac{N}{m}} \leq m \left(\frac{1}{e}\right)^{\frac{N}{m}}. \tag{3.3.11}$$

We show

$$m \left(\frac{1}{e}\right)^{\frac{N}{m}} \leq \frac{1}{m} \tag{3.3.12}$$

for $m = \left\lfloor \frac{1}{2} \cdot \frac{N}{\log N} \right\rfloor$, which is equivalent to the following inequalities

$$m \left(\frac{1}{e}\right)^{\frac{N}{m}} \leq \frac{1}{m} \Longleftrightarrow m^2 \leq e^{N/m} \Longleftrightarrow \log\left(m^2\right) \leq \log\left(e^{N/m}\right) \Longleftrightarrow 2m \log m \leq N.$$

The last inequality is true because

$$\begin{aligned} 2m \log m &= 2 \left\lfloor \frac{1}{2} \cdot \frac{N}{\log N} \right\rfloor \log \left\lfloor \frac{1}{2} \cdot \frac{N}{\log N} \right\rfloor \\ &\leq 2 \cdot \frac{1}{2} \cdot \frac{N}{\log N} \log \left(\frac{1}{2} \cdot \frac{N}{\log N}\right) \\ &\leq \frac{N}{\log N} \cdot \log N \\ &= N. \end{aligned}$$

Inequalities (3.3.10), (3.3.11) and (3.3.12) together yield

$$\mathbb{E} \inf_a D_\infty^*(a, \mathcal{P}) \leq 2 \cdot \frac{1}{\left\lfloor \frac{1}{2} \cdot \frac{N}{\log N} \right\rfloor} \leq 2 \cdot \frac{2}{\frac{1}{2} \cdot \frac{N}{\log N}} = 8 \cdot \frac{\log N}{N}.$$

□

3.3.2 Lower bound

Theorem 3.5. *There exists a constant $C > 0$ such that we have for the expectation of the optimally weighted star discrepancy in dimension $d = 1$ the lower bound*

$$\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) \geq C \cdot \frac{\log N}{N}.$$

Proof. We choose a rearrangement $t^\star_1, \ldots, t^\star_N$ of $t_1, \ldots, t_N$ with $t^\star_1 \leq t^\star_2 \leq \cdots \leq t^\star_N$ and rearrange $a_1, \ldots, a_N$ in the same way. Further, we set $t^\star_0 = 0$, $t^\star_{N+1} = 1$ and estimate (3.3.1) as

$$D^*_\infty(a, \mathcal{P}) \geq \sum_{j=1}^{i} a^\star_j - t^\star_i \text{ for all } i = 0, \ldots, N$$

and

$$D^*_\infty(a, \mathcal{P}) \geq t^\star_{i+1} - \sum_{j=1}^{i} a^\star_j \text{ for all } i = 0, \ldots, N.$$

Hence, we have

$$2D^*_\infty(a, \mathcal{P}) \geq t^\star_{i+1} - t^\star_i \text{ for all } i = 0, \ldots, N \text{ and all weights } a,$$

and therefore

$$\inf_a D^*_\infty(a, \mathcal{P}) \geq \frac{1}{2} \max_{i=0,\ldots,N} \left\{ t^\star_{i+1} - t^\star_i \right\}.$$

Choosing $\alpha = 0$ in Theorem 1.4 we get

$$\lim_{N \to \infty} \mathbb{P} \left(\max_{i=0,\ldots,N} \left\{ t^\star_{i+1} - t^\star_i \right\} \geq \frac{\log(N+1)}{N+1} \right) = 1 - \frac{1}{e}.$$

This means, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have

$$\mathbb{P} \left(\inf_a D^*_\infty(a, \mathcal{P}) \geq \frac{1}{2} \cdot \frac{\log(N+1)}{N+1} \right) \geq \frac{1}{2}$$

and therefore

$$\mathbb{P} \left(\inf_a D^*_\infty(a, \mathcal{P}) \geq c \cdot \frac{\log(N+1)}{N+1} \right) \geq \frac{1}{2}$$

for all $N \in \mathbb{N}$.

Now, we apply Markov's Inequality (Theorem 1.1) on

$$X = \inf_a D^*_\infty(a, \mathcal{P}),\ \varepsilon = c \cdot \frac{\log(N+1)}{N+1} \text{ and } f(\varepsilon) = \varepsilon.$$

This gives

$$\begin{aligned}
\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) &\geq c \cdot \frac{\log(N+1)}{N+1} \cdot \mathbb{P}\left(\inf_a D^*_\infty(a, \mathcal{P}) \geq c \cdot \frac{\log(N+1)}{N+1} \right) \\
&\geq c \cdot \frac{\log(N+1)}{N+1} \cdot \frac{1}{2} \\
&\geq \frac{c}{4} \cdot \frac{\log N}{N}.
\end{aligned}$$

□

Altogether, Corollary 3.3 and Theorem 3.5 yield the following theorem:

Theorem 3.6. *For the expectation of the optimally weighted star discrepancy in dimension $d = 1$ we have the asymptotic behavior*

$$\mathbb{E} \inf_a D^*_\infty(a, \mathcal{P}) \asymp \frac{\log N}{N}.$$

Unfortunately, the idea for the proof of Theorem 3.6 does not work for dimension $d > 1$ because we need Lemma 3.2 and therefor the possibility to arrange the points $t_1, \ldots t_N$ in ascending order. So, it is an open question how the expectation of the optimally weighted star discrepancy behaves in higher dimension.

4 Expectation of the optimally weighted L_2-discrepancy

4.1 Known results about the L_2-discrepancy

A celebrated milestone in discrepancy theory is the famous result of Roth, who showed the asymptotical lower bound for the L_2-discrepancy in 1954 [43]. The proof given by Roth is for $d = 2$ but he makes a remark how to generalize it for arbitrary dimension d. We reformulate Roth's result in the following theorem.

Theorem 4.1 (Roth)**.** *Let $N, d \in \mathbb{N}$ and $\mathcal{P} \subset [0,1]^d$ a set of N points. It holds*

$$D_2^*(\mathcal{P}) \geq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N} \tag{4.1.1}$$

with $c_d > 0$ only depending on d.

The main idea of Roth's proof was afterwards often used. In [5], Bilyk presents a nice, detailed discussion of Roth's method in a more modern notation.

The best known constant in (4.1.1) was found by Hinrichs and Markhasin ([34]). They show that Roth's lower bound holds with

$$c_d = \frac{7}{27 \cdot 2^{2d-1} \sqrt{(d-1)!}\,(\log 2)^{\frac{d-1}{2}}}.$$

From the monotonicity of the L_q norm it follows directly that

$$D_q^*(\mathcal{P}) \geq c_d \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

for all $q \geq 2$ and any point set $\mathcal{P} \subset [0,1]^d$ of N elements. For $1 < q < 2$, Schmidt showed in 1977

$$D_q^*(\mathcal{P}) \geq c_{d,q} \frac{(\log N)^{\frac{d-1}{2}}}{N} \tag{4.1.2}$$

for any point set $\mathcal{P} \subset [0,1]^d$ of N elements ([48]). Using Roth's technique, Chen proved the same asymptotical lower bound for the weighted L_2-discrepancy ([8, 9]).

In 1956, Davenport proved that Roth's bound (4.1.1) is best possible in dimension $d = 2$ ([14]). He found a symmetrized point set $\mathcal{P}$ with

$$D_2^*(\mathcal{P}) \leq c \, \frac{\sqrt{\log N}}{N}.$$

An alternative proof was given by Roth in 1976 ([44]), who used probabilistic methods. Another way of finding point sets with optimal order for the L_2-discrepancy is the so called digit scrambling, first done by Halton and Zaremba ([29]) and later picked up by Pillichshammer, Kritzer, Faure and others (see i.e. [23, 24, 37]).

In 1979, Roth proved that his lower bound is best possible in dimension $d = 3$ ([45]) and finally, generalized the proof to arbitrary dimension in 1980 ([46]). Also in 1980, Frolov found an alternative proof ([22]) for arbitrary dimension. The optimality of (4.1.2) for the L_q-discrepancy for every $q > 1$ was shown by Chen in 1980 ([7]).

All these upper bounds for dimension $d > 2$ are not constructive but show only the existence of point sets satisfying the upper bound. It remained an open problem in discrepancy theory for a long time to find explicitly given point sets in dimension $d > 2$ with optimal order in N for the L_2-discrepancy. Finally, in 2002 Chen and Skriganov constructed for the first time a point set $\mathcal{P}$ with

$$D_2^*(\mathcal{P}) \leq c_d \, \frac{(\log N)^{\frac{d-1}{2}}}{N}$$

in arbitrary dimension d ([10]). Subsequently, Skriganov extended the construction to the L_q-discrepancy for $1 \leq q < \infty$ ([49]).

A nice overview, in particular over many constructions of point sets with small L_2-discrepancy is given in [17].

In summary, the asymptotic behavior of the minimal (weighted) L_2-discrepancy in N is known and can be nicely written as

$$c_d \, \frac{(\log N)^{\frac{d-1}{2}}}{N} \leq \inf_{\substack{\mathcal{P} \subset [0,1]^d \\ \#\mathcal{P} = N}} \inf_{a \in \mathbb{R}^N} D_2^*(a, \mathcal{P}) \leq \inf_{\substack{\mathcal{P} \subset [0,1]^d \\ \#\mathcal{P} = N}} D_2^*(\mathcal{P}) \leq C_d \, \frac{(\log N)^{\frac{d-1}{2}}}{N} \tag{4.1.3}$$

(see [41]). An obvious question is which L_2-discrepancy random point sets yield. From Chapter 2 we know the famous result of Heinrich, Novak, Wasilkowski and Woźniakowski from 2001 ([30]). From (2.1.2) we directly see that

$$\mathbb{E} D_2^*(\mathcal{P})^2 = \mathrm{av}_2^*(N, d)^2 = C(1, 2, d) \, \frac{1}{N}.$$

Hence, random point sets yield worse order in N for the L_2-discrepancy than the optimal point sets. We ask ourselves now if we get better asymptotic behavior in N for the expected weighted L_2-discrepancy with optimal weights.

4.2 Expectation of the optimally weighted L_2-discrepancy for d=1

We want to compute

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$$

in the case $d = 1$.

Lemma 4.1. *In dimension $d = 1$ we have for the optimally weighted L_2-discrepancy of a point set $\mathcal{P} = \{t_1, \dots, t_N\}$, with $0 \le t_1 \le \dots \le t_N \le t_{N+1} = 1$, the equation*

$$\inf_a D_2^*(a, \mathcal{P})^2 = \frac{1}{3} t_1^3 + \frac{1}{12} \sum_{i=1}^{N} (t_{i+1} - t_i)^3. \tag{4.2.1}$$

Proof. We define for a point set $\mathcal{P} = \{t_1, \dots, t_N\}$, a sequence of weights $a = (a_j)_{j=1}^N$ and a function f the quadrature formula

$$Q_{N,d}(f, \mathcal{P}, a) = \sum_{j=1}^{N} a_j f(t_j).$$

Further, we denote by $e^{\mathbf{avg}}(Q_{n,d}(f, \mathcal{P}, a))$ the average case error of the quadrature formula $Q_{N,d}(f, \mathcal{P}, a)$ in the space $C([0,1]^d)$ with the norm $\|f\| = \max_{x \in [0,1]^d} |f(x)|$, equipped with the Wiener sheet measure. Woźniakowski shows in [54] that

$$D_2^*(a, \mathcal{P})^2 = e^{\mathbf{avg}}(Q_{N,1}(f, \widetilde{\mathcal{P}}, a))^2, \tag{4.2.2}$$

where $\widetilde{\mathcal{P}} = \left\{ \tilde{t}_j = 1 - t_j, j = 1, \dots, N \right\}$.

In [42], Ritter shows for a point set $\mathcal{P} = \{t_1, \dots, t_N\}$ with $0 = t_0 \le t_1 \le \dots \le t_N \le 1$ the equation

$$\inf_a e^{\mathbf{avg}}(Q_{N,1}(f, \mathcal{P}, a))^2 = \frac{1}{3}(1 - t_N)^3 + \frac{1}{12} \sum_{i=1}^{N} (t_i - t_{i-1})^3 \tag{4.2.3}$$

holds.

Combining (4.2.2) and (4.2.3) we get

$$\begin{aligned}\inf_a D_2^*(a,\mathcal{P})^2 &= \inf_a e^{\mathbf{avg}}(Q_{N,1}(f,\widetilde{\mathcal{P}},a))^2\\ &= \frac{1}{3}t_1^3 + \frac{1}{12}\sum_{i=1}^{N}((1-t_{N-i+1})-(1-t_{N-i+2}))^3\\ &= \frac{1}{3}(t_1)^3 + \frac{1}{12}\sum_{i=1}^{N}(t_{i+1}-t_i)^3,\end{aligned}$$

with $t_{N+1}=1$. □

In [42], Ritter shows further that the trapezoidal rule yields the optimal point sets for the quadrature formula

$$Q_{N,1}(f,\mathcal{P},a) = \sum_{j=1}^{N} a_j f(t_j).$$

Hence, because of (4.2.2), we have for the minimal optimally weighted L_2-discrepancy the following result.

Theorem 4.2. *For the minimal L_2-discrepancy in dimension $d=1$ equipped with the optimal weights holds*

$$\inf_{\substack{\mathcal{P}\subset[0,1]\\ \#\mathcal{P}=N}} \inf_a D_2^*(a,\mathcal{P})^2 = \frac{1}{3(2N+1)^2}. \tag{4.2.4}$$

Combining Lemma 4.1 and Theorem 1.3 we obtain the following theorem:

Theorem 4.3. *For the expectation of the optimally weighted L_2-discrepancy in dimension $d=1$ holds*

$$\mathbb{E}\inf_a D_2^*(a,\mathcal{P})^2 = \frac{3}{2(N+3)(N+2)(N+1)} + \frac{1}{2(N+3)(N+2)}. \tag{4.2.5}$$

Proof. We set $t_0=0$ and rewrite (4.2.1) as

$$\inf_a D_2^*(a,\mathcal{P})^2 = \frac{1}{4}t_1^3 + \frac{1}{12}\sum_{i=0}^{N}(t_{i+1}-t_i)^3 \tag{4.2.6}$$

and compute

$$\mathbb{E}\inf_a D_2^*(a,\mathcal{P})^2 = \frac{N!}{4}\int_Z t_1^3\,\mathrm{d}t + \frac{1}{12}\mathbb{E}W_N,$$

where

$$Z = \left\{ t \in \mathbb{R}^N : 0 \leq t_1 \leq \cdots \leq t_N \leq 1 \right\}$$

and

$$W_N = \sum_{j=0}^{N} (t_{i+1} - t_i)^3.$$

Computing the first summand gives

$$\begin{aligned}
\int_Z t_1^3 \,\mathrm{d}t &= \int_0^1 \int_0^{t_N} \ldots \int_0^{t_3} \int_0^{t_2} t_1^3 \,\mathrm{d}t_1 \,\mathrm{d}t_2 \ldots \mathrm{d}t_{N-1} \,\mathrm{d}t_N \\
&= \int_0^1 \int_0^{t_N} \ldots \int_0^{t_3} \frac{1}{4} t_2^4 \,\mathrm{d}t_2 \ldots \mathrm{d}t_{N-1} \,\mathrm{d}t_N \\
&= \int_0^1 \int_0^{t_N} \ldots \int_0^{t_4} \frac{1}{4 \cdot 5} t_3^5 \,\mathrm{d}t_3 \ldots \mathrm{d}t_{N-1} \,\mathrm{d}t_N \\
&= \int_0^1 \frac{6}{(N+2)!} t_N^{N+2} \,\mathrm{d}t_N \\
&= \frac{6}{(N+3)!}.
\end{aligned} \tag{4.2.7}$$

To compute the second summand we apply Theorem 1.3. We define $h : [0, \infty) \to \mathbb{R}$ with $h(x) = x^3$ on $[0, 1]$ and $\int_0^\infty h(r) \,\mathrm{d}r < \infty$. Hence, we get

$$\begin{aligned}
\mathbb{E} W_N &= N(N+1) \int_0^1 (1-r)^{N-1} r^3 \,\mathrm{d}r \\
&= N(N+1) \frac{6\Gamma(N)}{\Gamma(N+4)} \\
&= N(N+1) \frac{6(N-1)!}{(N-3)!} \\
&= \frac{6}{(N+2)(N+1)}.
\end{aligned} \tag{4.2.8}$$

Finally, equations (4.2.7) and (4.2.8) yield

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 = \frac{3}{2(N+3)(N+2)(N+1)} + \frac{1}{2(N+3)(N+2)}.$$

□

Comparing Theorem 4.3 to the result of Ritter in Theorem 4.2 we can state that the expectation of the optimally weighted L_2-discrepancy in dimension $d = 1$ has the same order in N as the minimal optimally weighted L_2-discrepancy.

Theorem 4.4. *For the expectation of the optimally weighted L_2-discrepancy in dimension $d = 1$ the following relation holds*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \inf_{\substack{\mathcal{P} \subset [0,1] \\ \#\mathcal{P} = N}} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{1}{N^2}. \tag{4.2.9}$$

In particular, comparing again Theorem 4.2 and Theorem 4.3, we have for the optimally weighted L_2-discrepancy that random points are worse than the optimal points only by the factor 6 .

4.3 Expectation of the optimally weighted L_2-discrepancy in higher dimension

How is the behavior of $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$ for $d > 1$? Because of the result achieved in the last section we hope for the asymptotical behavior $(\log N)^{d-1}/N^2$ in N.

Our above idea for $d = 1$ does not work for arbitrary $d > 0$ because we can not arrange the points $t_1, \ldots, t_N$ in ascending order and therefor not apply Lemma 4.1. For this reason, we present here an algorithm and a numerical simulation to compute $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$ for $d > 1$.

Let D be the numerical computed expected discrepancy $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$, hence $D \approx \mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$. To get D, we use direct simulation, which means

$$D = \frac{1}{m} \sum_{l=1}^{m} D(\mathcal{P}_l),$$

where m is the number of iterations in the direct simulation and

$$D(\mathcal{P}_l) = \inf_a D_2^*(a, \mathcal{P}_l)^2$$

for a fixed point set $\mathcal{P}_l$. The value $D_2^*(a, \mathcal{P}_l)^2$ can be computed in three steps:

First, we have to generate a pseudo-random point set $\mathcal{P}_l = \{t_1, \ldots t_N\}$.

Second, we have to find the optimal weights for this point set. To this end, we compute the local extrema $a_{\min}$ of $F(a) = D_2^*(a, \mathcal{P}_l)^2$. We use the representation of $D_2^*(a, \mathcal{P}_l)^2$ given by Lemma 1.10, namely

$$F(a) = D_2^*(a, P_l)^2 = \frac{1}{3^d} - \frac{1}{2^{d-1}} \sum_{j=1}^{N} a_j \prod_{k=1}^{d} \left(1 - t_{j,k}^2\right) + \sum_{i,j=1}^{N} a_i a_j \prod_{k=1}^{d} \left(1 - \max\{t_{i,k}, t_{j,k}\}\right).$$

Differentiating with respect to a_j gives

$$\frac{\partial F}{\partial a_j} = -\frac{1}{2^{d-1}} \prod_{k=1}^{d} (1 - t_{j,k}^2) + 2 \sum_{i=1}^{N} a_i \prod_{k=1}^{d} (1 - \max\{t_{i,k}, t_{j,k}\}) .$$

To find the minimal a we set $\frac{\partial F}{\partial a_j} = 0$ and get a system of linear equations $Aa = b$, with $A \in \mathbb{R}^{N \times N}$ given as

$$A(i,j) = 2 \prod_{k=1}^{d} (1 - \max\{t_{i,k}, t_{j,k}\})$$

and $b \in \mathbb{R}^N$ given as

$$b(j) = \frac{1}{2^{d-1}} \prod_{k=1}^{d} (1 - t_{j,k}^2).$$

The solution of this system is $a_{\min}$.

Third, we have to compute $D_2^*(a_{\min}, \mathcal{P}_l)^2$, which is

$$D_2^*(a_{\min}, \mathcal{P}_l)^2 = \frac{1}{3^d} - \sum_{j=1}^{N} a_{\min}(j) b(j) + \sum_{i,j=1}^{N} \frac{1}{2} a_{\min}(j) a_{\min}(i) A(i,j).$$

The following C code has been used for the direct simulation of $\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$.

The main function has no input parameters and, for fixed d, returns a table with the number of points N in the left column and the numerical simulated expected discrepancy $D \approx \mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2$ in the right column. The dimension d can be changed in the main function.

```
// Program header
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#include <math.h>

// declaration of used functions
double ex_opt_l2_discr(int N, int m, int d);
double *get_rand_matrix_dxN(int d, int N);
double my_rand(void);
double opt_l2_discr(double *T, int N, int d);
double max(double a, double b);
```

```
int main(int argc, char *argv[]) {
  // initialization of the random number generator
  time_t rand_init;
  time(&rand_init);
  srand((unsigned int)rand_init);

  // simulation parameters
  int N_start = 10;
  int N_end   = 10000;
  int N_delta = 10;
  int d = 2;
  int m = 10000;

  int N;
  for (N = N_start; N <= N_end; N += N_delta) {
    // sets the step size in dependence of N
    if (N >= 100 && N < 1000)
      N_delta = 100;
    else if (N >= 1000)
      N_delta = 1000;

    // computes expectation of the optimally weighted
    // L_2-discrepancy of N points in dimension d
    // via direct simulation with m steps
    double E = ex_opt_l2_discr(N,m,d);

    // prints the result
    printf("%d \t %.15e\n",N,E);
  }
  return 0;
}
```

The following function computes and returns the numerical simulated expectation of the optimally weighted L_2-discrepancy of N points in dimension d via direct simulation. The input parameters are the number of points N, the dimension d and the number of simulation steps m in the direct simulation.

```
double ex_opt_l2_discr(int N, int m, int d) {
  // initialization of expectation with 0
  double E = 0;
```

```
  // computes m times the optimally weighted L_2-discrepancy
  // of a point set saved in T and computes the average
  int i;
  for (i = 0; i < m; i++) {
    // generates a random dxN-matrix T
    double *T = get_rand_matrix_dxN(d,N);

    // computes optimally weighted L_2-discrepancy of T
    double disc = opt_l2_discr(T,N,d);

    // computes new average
    E = 1.0/(i+1)*(E*i+disc);

    // frees the memory
    free(T);
  }

  // returns the average
  return E;
}
```

The following functions return a random Matrix of dimension $d \times N$. The elements are independent and uniformly distributed random numbers in $[0, 1]$. Input parameters are the dimensions of the matrix N and d.

```
double *get_rand_matrix_dxN(int d, int N) {
  // allocates memory
  double * T = calloc(d*N, sizeof(double));

  // fills matrix T with independent and uniformly
  // distributed random numbers in [0,1]
  int i;
  for (i = 0;i < d*N;i++) {
    T[i] = my_rand();
  }

  // returns matrix
  return T;
}

double my_rand() {
```

```
    // computes and returns an uniformly distributed
    // random number in [0,1]
    return (double)rand() / (double)RAND_MAX;
}
```

The following function finds the optimal weights for a point set $\mathcal{P} = \{t_1, \ldots, t_N\}$, saved in matrix T, and computes and returns the optimally weighted L_2-discrepancy of this point set. Input parameters are the matrix T, the number of points N and the dimension d.

```
double opt_l2_discr(double *T, int N, int d) {
    // initialization of optimally weighted L_2-discrepancy
    double disc = 1./pow(3,d);

    // allocates memory
    double *A     = calloc(N*N,sizeof(double));
    double *Atemp = calloc(N*N,sizeof(double));
    double *b     = calloc(N   ,sizeof(double));
    double *x     = calloc(N   ,sizeof(double));
    int *pivot    = calloc(N   ,sizeof(double));

    int n1 = N;
    int n2 = 1;
    int ok;

    // defines the matrix A and the vector b for the
    // system of linear equations Ax = b
    // solution x are the optimal weights for
    // the point set T
    int i,j,k;
    for (i = 0; i < N; i++) {
        b[i] = 1./pow(2,d-1);
        for (k = 0; k < d; k++){
            b[i] = b[i]*(1-T[k*N+i]*T[k*N+i]);
        }
        for (j = 0; j < N; j++) {
            A[i*N+j] = 2.0;
            for (k = 0; k < d; k++) {
                A[i*N+j] = A[i*N+j]*(1-max(T[k*N+j],T[k*N+i]));
            }
        }
```

```
  }

  // makes a copy of A and b for later use
  // because Atemp and x will be overwritten
  for (i = 0; i < N; i++) {
    x[i] = b[i];
  }
  for (i = 0; i < N*N; i++) {
    Atemp[i] = A[i];
  }

  // solves Atemp*a = x with LU-factorization with pivoting
  // and saves the result a in x
  // Using lapack to solve Atemp*a = x
  dgesv_(&n1, &n2, Atemp, &n1, pivot, x, &n1, &ok);

  // computes optimally weighted L_2-discrepancy of
  // point set T with optimal weights x
  for (i = 0; i < N; i++) {
    disc += -x[i]*b[i];
    for (j = 0; j < N; j++) {
      disc += 0.5*x[i]*x[j]*A[i*N+j];
    }
  }

  // frees memory
  free(Atemp);
  free(A);
  free(b);
  free(x);
  free(pivot);
  return disc;
}

// returns the maximum of two numbers
double max(double a, double b) {
  return a >= b ? a:b;
}
```

We used $m = 10.000$ steps for the direct simulation. The number of points was chosen between $N = 100$ and $N = 10.000$ and the dimension between $d = 2$ and $d = 5$. It is possible to get results for higher dimension d though the execution time would increase significantly.

After we made the simulation, we fitted the function $c\,(\log N)^{d-1}/N^2$ on our data and found an optimal constant $c(d)$ for each dimension $d = 2, 3, 4, 5$. The result of section 4.2 gives us reason to hope for this asymptotic behavior in N.

The following pictures illustrate the results of the simulation for each dimension $d = 2, 3, 4, 5$ and the corresponding fit functions $E = c(d)\,(\log N)^{d-1}/N^2$.

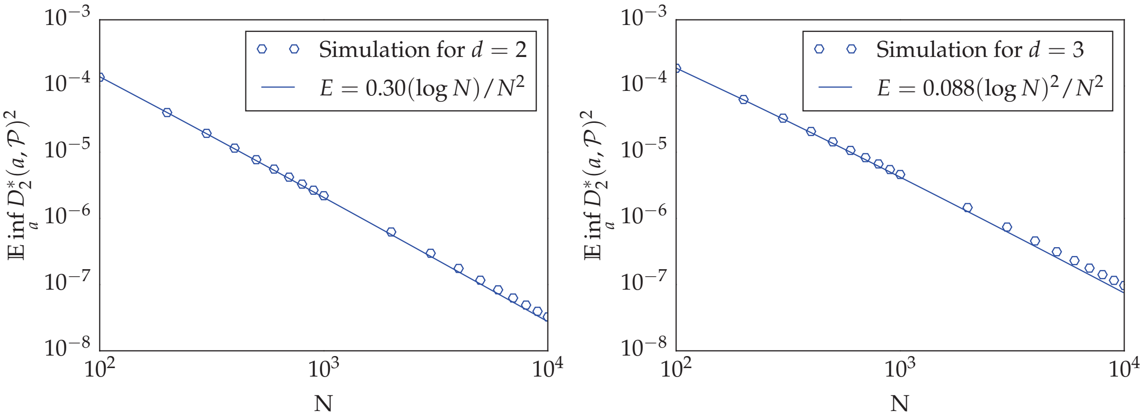

Figure 4.1: Simulation of the expectation of the optimally weighted L_2-discrepancy in dimension $d = 2$ and $d = 3$

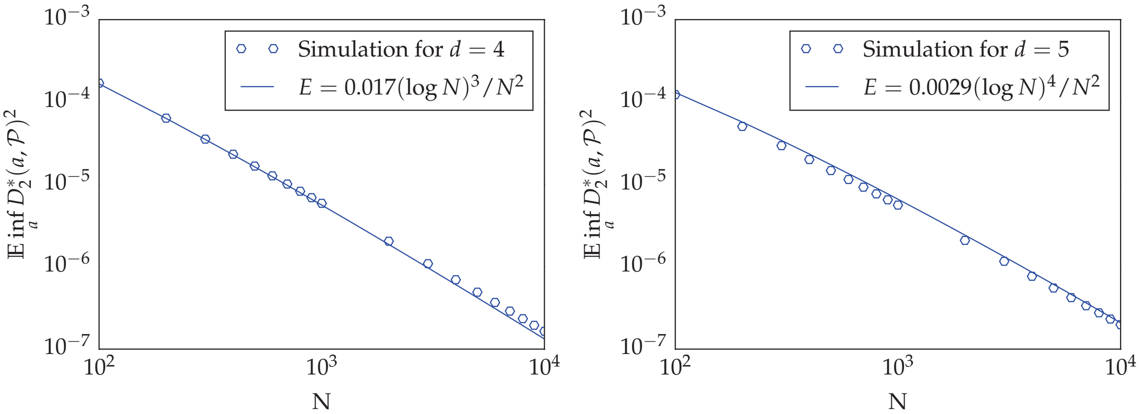

Figure 4.2: Simulation of the expectation of the optimally weighted L_2-discrepancy in dimension $d = 4$ and $d = 5$

Indeed, the result $D \approx \mathbb{E}\inf_a D_2^*(a, \mathcal{P})^2$ of the simulation seems to have the asymptotic behavior $(\log N)^{d-1}/N^2$ in N.

Consequently, the numerical simulations support the following conjecture.

Conjecture 4.1. *For the expectation of the optimally weighted L_2-discrepancy in arbitrary dimension d the following relation holds*

$$\mathbb{E} \inf_a D_2^*(a, \mathcal{P})^2 \asymp \frac{(\log N)^{d-1}}{N^2}.$$

Comparing this to (4.1.3) this conjecture would involve that the expected weighted L_2-discrepancy of a random point set equipped with the optimal weights has the same asymptotic behavior in N as the (unweighted) L_2-discrepancy of the optimal point set.

Bibliography

[1] C. Aistleitner: *Covering numbers, dyadic chaining and discrepancy*, J. Complexity **27**, 531-540 (2011)

[2] C. Aistleitner, M. Hofer: *Probabilistic Discrepancy bound for Monte Carlo Point sets*, Mathematics of Computation **83**, 1373-1381 (2013)

[3] T. van Aardenne-Ehrenfest: *Proof of the impossibility of a just distribution of an infinite sequence of points over an interval*, Proc. Kon. Ned. Akad. v. Wetensch. **48**, 266-271 (1945)

[4] J. Beck, W.W.L. Chen: *Irregularities of distribution* (Cambridge University Press, Cambridge, 1987)

[5] D. Bilyk: *Roth's orthogonal function method in discrepancy theory and some new connections*, in A Panorama of Discrepancy Theory, ed. by W.W.L. Chen, A. Srivastav, G. Travaglini, Lecture Notes in Mathematics 2107 (Springer, Cham/ Heidelberg/ New York/ Dordrecht/ London, 2014)

[6] D. Bilyk, M.T. Lacey, A. Vagharshakyan:*On the small ball inequality in all dimensions*, J. Funct. Anal. **254**, 2470-2502 (2008)

[7] W.W.L. Chen: *On irregularities of point distribution*, Mathematika **27**, 153-170 (1980)

[8] W.W.L. Chen: *On irregularities of distribution and approximate evaluation of certain functions*, Quarterly J. Math. Oxford **36**, 173-182 (1985)

[9] W.W.L. Chen: *On irregularities of distribution and approximate evaluation of certain functions* II, in Analytic Number Theory and Diophantine Problems, ed. by A.C. Adolphson, A. Ghosh, J.B. Conrey, R.I. Jager, Proceedings of a Conference at Oklahoma State University, 1984 (Birkhäuser, Basel, 1987)

[10] W.W.L. Chen, M.M. Skriganov: *Explicit constructions in the classical mean squares problem in irregularities of point distribution*, J. Reine Angew. Math. **545**, 67-95 (2002)

[11] J.G. van der Corput: *Verteilungsfunktionen* I, Proc. Kon. Ned. Akad. v. Wetensch. **38**, 813-821 (1935)

[12] J.G. van der Corput: *Verteilungsfunktionen* II, Proc. Kon. Ned. Akad. v. Wetensch. **38**, 1058-1066 (1935)

[13] D.A. Darling: *On a Class of Problems Related to the Random Division of an Interval*, The Annals of Mathematical Statistics **24**, 239-253 (1953)

[14] H. Davenport: *Note on irregularities of distribution*, Mathematika **3**, 131-135 (1956)

[15] J. Dick, A. Hinrichs, F. Pillichshammer: *Proof techniques in quasi-Monte Carlo theory*, J. Complexity **31**, 327-371 (2015)

[16] J. Dick, F. Pillichshammer: *Digital nets and sequences. Discrepancy Theory and Quasi-Monte Carlo Integration* (Cambridge University Press, Cambridge, 2010)

[17] J. Dick, F. Pillichshammer: *Explicit constructions of point sets and sequences with low discrepancy*, in Uniform Distribution and Quasi-Monte Carlo Methods, ed. by P. Kritzer, H. Niederreiter, F. Pillichshammer, A. Winterhof, Radon Series on Computational and Applied Mathematics 15 (de Gruyter, Berlin, 2014)

[18] B. Doerr: *A lower bound for the discrepancy of a random point set*, J. Complexity **30**, 16-20 (2014)

[19] B. Doerr, M. Gnewuch, P. Kritzer, F. Pillichshammer: *Component-by-component construction of low-discrepancy point sets of small size*, Monte Carlo Methods Appl. **14**, 129-149 (2008)

[20] B. Doerr, M. Gnewuch, M. Wahlström: *Algorithmic construction of low-discrepancy point sets via dependent randomized rounding*, J. Complexity **26**, 490-507 (2010)

[21] M. Drmota, R.F. Tichy: *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics, vol. 1651 (Springer, Berlin and Heidelberg, 1997)

[22] K.K. Frolov: *Upper bounds on the discrepancy in* $L_p, 2 \leq p < \infty$, Dokl. Akad. Nauk USSR **252**, 805-807 (1980)

[23] H. Faure, F. Pillichshammer: L_2 *discrepancy of two-dimensional digitally shifted Hammersley point sets in base* b, in Monte Carlo and Quasi-Monte Carlo Methods 2008, ed. by P. L'Ecuyer, A. Owen (Springer, Berlin, 2009)

[24] H. Faure, F. Pillichshammer, G. Pirsic: L_2 *discrepancy of linearly digit scrambled Zaremba point sets*, Unif. Distrib. Theory **6**, 59-81 (2011)

[25] M. Gnewuch: *Bounds for the average* L_p*-extreme and* L_∞*-extreme dis-*

crepancy, Electron. J. Combin. **12**, 1-11 (2005), Research Paper 54

[26] M. Gnewuch: *Weighted geometric discrepancies and numerical integration on reproducing kernel Hilbert spaces*, J. Complexity **28**, 2-17 (2012)

[27] M. Gräf, D. Potts, G. Steidl: *Quadrature rules, discrepancies and their relations to halftoning on the torus and the sphere*, SIAM J. Sci. Comput. **34**, A2760-A2791 (2012)

[28] R.L. Graham, D.E. Knuth, O. Patashnik: *Concrete Mathematics: A Foundation for Computer Science, 2nd ed.* (Addison-Wesley, Reading, MA, 1994)

[29] J.H. Halton, S.K. Zaremba: *The extreme and L^2 discrepancies of some plane sets*, Monatsh. Math. **73**, 316-328 (1969)

[30] S. Heinrich, E. Novak, G. Wasilkowski, H. Woźniakowski: *The inverse of the star-discrepancy depends linearly on the dimension*, Acta Arith. **96**, 279-302 (2001)

[31] A. Hinrichs, E. Novak: *New bounds for the star discrepancy*, Extended abstract of a talk at the Oberwolfach seminar "Discrepancy Theory and its Applications", Report No. 13/2004, Mathematisches Forschungsinstitut Oberwolfach

[32] A. Hinrichs: *Covering numbers, Vapnik-Červonenkis classes and bounds for the star-discrepancy*, J. Complexity **20**, 477-483 (2004)

[33] A. Hinrichs: *Discrepancy, Integration and Tractability*, in Monte Carlo and Quasi-Monte Carlo Methods 2012, ed. by J. Dick, F.Y. Kuo, G.W. Peters, I.H. Sloan, Springer Proceedings in Mathematics & Statistics Volume 65 (Springer, Berlin and Heidelberg, 2013)

[34] A. Hinrichs, L. Markhasin: *On lower bounds for the L_2-discrepancy*, J. Complexity **27**, 127-132 (2011)

[35] A. Hinrichs, H. Weyhausen: *Asymptotic behavior of average L_p-discrepancies*, J. Complexity **28**, 425-439 (2012)

[36] A. Klenke: *Wahrscheinlichkeitstheorie* (Springer, Berlin, 2006)

[37] P. Kritzer, F. Pillichshammer: *An exact formula for the L_2 discrepancy of the shifted Hammersley point set*, Unif. Distrib. Theory **1**, 1-13 (2006)

[38] M. Ledoux, M. Talagrand: *Probability in Banach spaces* (Springer, Berlin, 1999)

[39] J. Matoušek: *Geometric discrepancy. An illustrated guide* (Springer, Berlin, 1999)

[40] H. Niederreiter: *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 63 (SIAM, Philadelphia, PA, 1992)

[41] E. Novak, H. Woźniakowski: *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals*, EMS Tracts in Mathematics, Vol. 12 (Eur. Math. Soc., Zürich 2010)

[42] K. Ritter: *Average-Case Analysis of Numerical Problems*, Lecture Notes in Math. 1733 (Springer, Berlin, 2000)

[43] K.F. Roth: *On irregularities of distributions*, Mathematika **1**, 73-79 (1954)

[44] K.F. Roth: *On irregularities of distributions* II, Comm. Pure Appl. Math. **29**, 739-744 (1976)

[45] K.F. Roth: *On irregularities of distributions* III, Acta Arithm. **35**, 373-384 (1979)

[46] Roth K.F.: *On irregularities of distributions* IV, Acta Arithm. **37**, 67-75 (1980)

[47] W.M. Schmidt: *Irregularities of distribution* VII, Acta Arith. **21**, 45-50 (1972)

[48] W.M. Schmidt: *Irregularities of distribution* X (Academic Press, New York, 1977)

[49] M.M. Skriganov: *Harmonic analysis on totally disconnected groups and irregularities of point distributions*, J. Reine Angew. Math. **600**, 25-49 (2006)

[50] S. Steinerberger: *The asymptotic behavior of the average L_p-discrepancies and a randomized discrepancy*, Electron. J. Combin. **17**, 1-18 (2010) Research Paper 106

[51] J.F. Traub, A.G. Werschulz: *Complexity and Information* (Cambridge University Press, Cambridge, 1998)

[52] T.T. Warnock: *Computational investigations of low discrepancy point sets*, in Applications of Number Theory to Numerical Analysis, ed. by S.K. Zaremba, (Academic Press, New York, 1972)

[53] H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77**, 313-352 (1916)

[54] H. Woźniakowski: *Average case complexity of multivariate integration*, Bull. Amer. Math. Soc. **24**, 185-191 (1991)

List of Figures

Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt habe, keine Textabschnitte oder Ergebnisse eines Dritten oder eigenen Prüfungsarbeiten ohne Kennzeichnung übernommen und alle von mir benutzten Hilfsmittel, persönliche Mitteilungen und Quellen in meiner Arbeit angegeben habe,
- dass ich die Hilfe eines Promotionsberaters nicht in Anspruch genommen habe, und das Dritte weder unmittelbar noch mittelbar geldwerte Leistungen von mir für Arbeiten erhalten haben, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen,
- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder andere wissenschaftliche Prüfung eingereicht habe,
- dass ich die gleiche oder eine in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung bei keiner anderen Hochschule als Dissertation eingereicht habe.

Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts haben mich folgende Personen unterstützt:
Prof. Aicke Hinrichs, Johannes Kepler Universität Linz

Jena, den 02.07.2015

Lebenslauf

Persönliche Daten

Name:	Heidi Weyhausen
Geburtstag:	23.09.1986
Geburtsort:	Saalfeld
Familienstand:	verheiratet
Kinder:	ein Sohn (geboren am 17.02.2013)

Schulbildung

1997 - 2005	Heinrich-Böll-Gymnasium Saalfeld Abschluss: Abitur (Note: 1,1)

Hochschulbildung

2005 - 2010	Studium der Mathematik mit Nebenfach Physik an der Friedrich-Schiller-Universität Jena Abschluss: Diplom (Prädikat: Mit Auszeichung)

Berufliche Tätigkeit

seit 10/2010	Wissenschaftlicher Mitarbeiter am Institut für Mathematik der Friedrich-Schiller-Universität Jena
01/2013 - 10/2013	Unterbrechung der Arbeit als wissenschaftlicher Mitarbeiter aufgrund von Mutterschutz und Elternzeit

Jena, den 02.07.2015